A positive solution on $\mathbb{R}^N$ to a system of elliptic equations of FitzHugh-Nagumo type

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1 Introduction

In this paper we consider the following system of semilinear elliptic equations

$$\begin{cases} -\Delta u = f(u) - v & \text{in } \mathbb{R}^N, \\ -\Delta v = \delta (u - \gamma v) & \text{in } \mathbb{R}^N, \\ u(x) \to 0, & \text{if } |x| \to \infty, \\ v(x) \to 0, & \text{if } |x| \to \infty, \end{cases} \tag{1}$$

where $\delta, \gamma > 0$ and $f(u) = -u(u-1)(u-a)$ with $0 < a < 1/2$.

Klaasen and Troy, [12] investigated this system when $N = 1$. Using a shooting argument they proved under certain conditions on the parameters the existence of a nontrivial solution. They also proved the existence an infinite number of periodic solutions to the equations in (1). On smooth bounded domains, Klaasen and Mitidieri, [11] studied the corresponding system, subjected to homogeneous Dirichlet boundary conditions. Their results show that both the domain and the parameters play a role in the existence and nonexistence of solutions. Results about the positivity of these solutions were obtained by De Figueiredo and Mitidieri, [6].

With a suitable rescaling, solutions on a bounded domain, say Ω, are seen to be steady state solutions of the system

$$\begin{cases} u_t = D_1 \Delta u + f(u) - v & \text{in } (0,\infty) \times \Omega, \\ v_t = D_2 \Delta v + \varepsilon (u - \gamma v) & \text{in } (0,\infty) \times \Omega, \end{cases} \tag{2}$$

with $D_1, D_2, \varepsilon > 0$. With $D_2 = 0$, (2) is known as the FitzHugh-Nagumo equations. These equations are used as a model for nerve conduction and other chemical and biological systems, [7], [10].

$>$From here on we will mean by a solution a pair $(u, v) \in C^\infty(\mathbb{R}^N) \times C^\infty(\mathbb{R}^N)$ that solves system (1). In this paper solutions are constructed by using solutions to the system corresponding to (1) on the ball B_R, and letting $R \to \infty$. This method of constructing solutions on $\mathbb{R}^N$ was also used for example by Berestycki and Lions [3] and Ni [15] in the scalar case, i.e. $\delta = 0$. Klaasen and Mitidieri [11] proved the existence of solutions on bounded domains using variational methods. In order to do so they used an a priori bound for the maxima of the solutions. We have a different approach of finding solutions on the ball B_R. Using a transformation similar to the one used by Mancini and Mitidieri in [13], and a modification of f, we obtain a quasimonotone system. This approach imposes the first condition on the parameters. Solutions of this system, in an appropriate range, can be used to obtain solutions to the original system. The existence of solutions to the quasimonotone system follows from the existence of pairs of super- and subsolutions and a multiplicity result due to Amann, [2].

We remark that the conditions on γ and δ which we assume are stronger than those imposed [11] and [6] for both the existence and positivity of solutions on the ball. However by using the quasimonotone method we obtain a sharper uniform upper bound as well as a uniform lower bound for the maxima. Also, by a result of Troy, [17], we have that the solutions are radially symmetric and decreasing. The combination of these properties will enable us to show that the solution on $\mathbb{R}^N$ is nontrivial and tends to zero as $R \to \infty$.

In Section 2 we state our conditions on the parameters as well as the main result. In Section 3 we prove some auxiliary results for quasimonotone systems. The main result is proven in Section 4.

2 Main result

Before we state our conditions on the parameters δ and γ we give some known results for certain ranges of these parameters. For the problem

$$\begin{cases} -\Delta u &= f(u) - v & \text{in } B_R, \\ -\Delta v &= \delta(u - \gamma v) & \text{in } B_R, \\ \quad u &= v = 0 & \text{on } \partial B_R, \end{cases} \tag{3}$$

the following holds.

- If $\gamma < 4/(1-a)^2$ then (3) has only the trivial solution, Klaasen and Mitidieri, [11].
- If $\gamma > 9/(2a^2 - 5a + 2)$ and $R > 0$ is large enough then (3) has two nontrivial solutions, Klaasen and Mitidieri, [11].
- If $1/\gamma < a < \delta\gamma - 2\sqrt{\delta}$ then all solutions to (3) are positive, De Figueiredo and Mitidieri [6].

For (1) with $N = 1$ the following holds, Klaasen and Troy [17].

- If $\gamma > \max\left\{9/(2a^2 - 5a + 2), 2/\sqrt{\delta} + (1-a)/\delta\right\}$ then there exists a nonconstant solution with $u'(0) = v'(0) = 0$ and $u(x), v(x) \to 0$ if $|x| \to \infty$ as well as an infinite number of periodic solutions.
- If $\max\left\{4/(1-a)^2, 2/\sqrt{\delta} + (1-a)/\delta\right\} < \gamma < 9/(2a^2 - 5a + 2)$ then there exists a nonconstant solution with $u'(0) = v'(0) = 0$ and $(u(x), v(x)) \to (\sigma_\gamma, \sigma_\gamma/\gamma)$ if $|x| \to \infty$, (see figure 1), as well as an infinite number of periodic solutions.

Our first condition is

Condition 1 $\delta\gamma - 2\sqrt{\delta} \geq 1 - a$.

We will use this condition to construct a quasimonotone system. As is known, quasimonotone systems share many properties with the scalar case. In particular, using sub- and supersolutions we can find solutions to the quasimonotone system and use these solutions to get solutions to the original system.

In order to find a subsolution to the quasimonotone system, we impose the following condition.

Condition 2 $\beta := \frac{1}{2\delta}(\delta\gamma + a - 1) + \frac{1}{2\delta}\sqrt{(\delta\gamma + a - 1)^2 - 4\delta} > \dfrac{9}{2a^2 - 5a + 2}.$

By condition 1 one has that $\beta \in \mathbb{R}$. A direct calculation shows that $f(u) - \frac{1}{\beta}u = 0$ has two nonzero roots, say $0 < \theta_\beta < \sigma_\beta$, if and only if $\beta > 4/(a^2 - 2a + 1)$. Since $9/(2a^2 - 5a + 2) > 4/(a^2 - 2a + 1)$, condition 2 implies that this holds. In fact, condition 2 is equivalent with

$$\int_0^{\sigma_\beta} \left(f(s) - \frac{1}{\beta}s \right) ds > 0, \tag{4}$$

see [11]. If (4) holds then the single equation $-\Delta u = f(u) - \frac{1}{\beta}u$ in Ω, $u = 0$ on $\partial\Omega$ has a nontrivial solution, [3] and [5]. This function will be used to obtain a subsolution for the quasimonotone system.

Since $\gamma > \beta$ the equation $f(u) - \frac{1}{\gamma}u = 0$ also has two nonzero roots, say $0 < \theta_\gamma < \sigma_\gamma$, with $0 < \theta_\gamma < \theta_\beta < \sigma_\beta < \sigma_\gamma$, and

$$\int_0^{\sigma_\gamma} \left(f(s) - \frac{1}{\gamma}s \right) ds > 0.$$

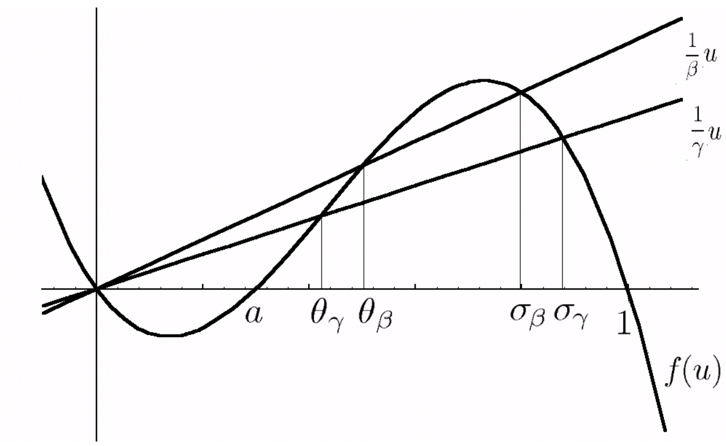

Figure 1

We remark that our conditions on the parameters are more restrictive than conditions of Klaasen and Troy for the system with $N = 1$, as can be seen by the fact that conditions 1 and 2 can be combined as follows:

$$\gamma > \max\left\{2/\sqrt{\delta} + (1-a)/\delta, 9/\left(2a^2 - 5a + 2\right) + \left(2a^2 - 14a + 11\right)/(9\delta)\right\}.$$

Our main result is:

Theorem 1 *If conditions 1 and 2 hold then there exists a positive solution $(u,v) \in C^\infty(\mathbb{R}^N) \times C^\infty(\mathbb{R}^N)$ to system (1). Moreover the functions u and v are radially symmetric, decreasing and satisfy*

$$\theta_\gamma < \max_{x\in\mathbb{R}^N} u(x) < \sigma_\gamma, \qquad \textit{and} \qquad \max_{x\in\mathbb{R}^N} v(x) < \frac{1}{\gamma}\sigma_\gamma$$

3 Auxiliary results for quasimonotone systems.

First we introduce some notation. Let $\Omega \subset \mathbb{R}^N$ be open and bounded with C^3 boundary $\partial\Omega$. By $\mathbf{u} = (u_1, \ldots, u_M)$ we will denote elements of $C\left(\bar{\Omega}\right)^M$. We consider the usual ordering of $C\left(\bar{\Omega}\right)^M$, i.e.

$$\mathbf{u} \leq \mathbf{v} \text{ if } u_i(x) \leq v_i(x) \text{ for } i = 1, \ldots, M \text{ and all } x \in \Omega.$$

We also write

$$\mathbf{u} < \mathbf{v} \text{ if } \mathbf{u} \leq \mathbf{v} \text{ and } \mathbf{u} \neq \mathbf{v},$$

and

$$\mathbf{u} \ll \mathbf{v} \text{ if } u_i(x) < v_i(x) \text{ for } i = 1, \ldots, M \text{ and all } x \in \Omega.$$

If $\mathbf{u} \leq \mathbf{v}$ we denote by $[\mathbf{u}, \mathbf{v}]$ the order interval $[\mathbf{u}, \mathbf{v}] := \{\mathbf{w}\,;\, \mathbf{u} \leq \mathbf{w} \leq \mathbf{v}\} \subset C\left(\bar{\Omega}\right)^M$. The set of nonnegative functions in $C_0^\infty(\Omega)$ will be denoted by $\mathcal{D}^+(\Omega)$ and we write $\mathcal{D}^{+,M}(\Omega)$ for the product $(\mathcal{D}^+(\Omega))^M$.

In this section we will consider the system

$$\begin{cases} -\Delta\mathbf{u} = F(\mathbf{u}) & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{5}$$

with $F(\cdot) = (F_1(\cdot), \ldots, F_M(\cdot))$. We assume that F_i differentiable and that

$$\left|\frac{\partial F_i}{\partial x_i}\right| \leq K$$

for some $K > 0$.

Definition 2 *System (5) is called **quasimonotone** if*

$$\frac{\partial F_i}{\partial x_j} \geq 0 \textit{ for } i \neq j \textit{ on } \Omega. \tag{6}$$

Definition 3 *We call* $\mathbf{u} \in C(\bar{\Omega})^M$ *a* ***supersolution*** *for system (5) if*

$$\int_\Omega \mathbf{u} \cdot (-\Delta\varphi)\ dx \geq \int_\Omega F(\mathbf{u}) \cdot \varphi\, dx \qquad \text{for every } \varphi \in \mathcal{D}^{+,M}(\Omega)$$

and

$$\mathbf{u} \geq \mathbf{0} \qquad \text{on } \partial\Omega.$$

A supersolution is ***strict*** *if there exists a* $\varphi \in \mathcal{D}^{+,M}(\Omega)$ *such that*

$$\int_\Omega \mathbf{u} \cdot (-\Delta\varphi)\ dx > \int_\Omega F(\mathbf{u}) \cdot \varphi\, dx,$$

or if

$$\mathbf{u} > \mathbf{0} \qquad \text{on } \partial\Omega.$$

Subsolutions *are defined by reversing the inequality signs.*

Remark: Recall, see [5, Lemma A.1.], that if $\omega \geq 0$ and $u \in C(\bar{\Omega})$ satisfies $u \geq 0$ on $\partial\Omega$, and

$$\int_\Omega (u(-\Delta\varphi) + \omega u\varphi)\ dx \geq 0 \qquad \text{for every } \varphi \in \mathcal{D}^{+}(\Omega),$$

then $u \geq 0$ in Ω. Moreover, if for at least one $\varphi \in \mathcal{D}^+(\Omega)$ the inequality above becomes strict, then there exists $\alpha > 0$ such that $u \gg \alpha\psi$ with ψ the principle eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions, [4, Corollary in Appendix]. The function ψ can also be replaced by any other function $\psi_1 \in C^1\left(\bar{\Omega}\right)$ with $\psi_1 \gg 0$ and $\frac{\partial\psi_1}{\partial\nu} < 0$ on $\partial\Omega$.

We also recall some properties of the solution operators. For details we refer to [9]. For every $g \in L^p(\Omega)$, $p > N/2$, and $\omega \geq 0$, the Dirichlet problem, $(-\Delta + \omega)u = g$ in Ω, $u = 0$ on $\partial\Omega$ has a unique solution $u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\bar{\Omega})$ which we shall denote by $u = K_\omega g$. One has that K_ω maps $C(\bar{\Omega})$ compactly into $C^{1,\alpha}(\bar{\Omega})$ for $\alpha \in [0,1)$. Indeed, since for $p > N$ it holds that $W^{2,p}(\Omega)$ is compactly imbedded in $C^{1,\alpha}(\bar{\Omega})$ with $0 < \alpha < (1 - N/p)$, we have that $K_\omega : C(\bar{\Omega}) \subset L^p(\Omega) \to C^{1,\alpha}(\bar{\Omega})$ is a compact mapping. We shall also use the notation $F_{\omega,i}(\mathbf{u}) = F_i(\mathbf{u}) + \omega u_i$, $i = 1, \ldots, M$. The mappings $F_{\omega,i}$ is continuous from $C(\bar{\Omega})^M$ to $C(\bar{\Omega})$. Consequently the operator $T_\omega : C(\bar{\Omega})^M \to \left(C^{1,\alpha}(\bar{\Omega})\right)^M$ defined by

$$T_\omega(\mathbf{u}) := (K_\omega(F_{\omega,1}(\mathbf{u})), \ldots, K_\omega(F_{\omega,M}(\mathbf{u}))). \tag{7}$$

is continuous and compact. We prove in the following lemma that T_ω is increasing for ω large enough.

Lemma 4 *Choose $\omega > 0$ such that*

$$\frac{\partial F_i}{\partial u_i} + \omega > 0 \qquad \text{on } \mathbb{R}^M \text{ for } i = 1, \ldots, M. \tag{8}$$

1. *If $\mathbf{u}, \mathbf{v} \in C(\bar{\Omega})^M$ and $\mathbf{u} > \mathbf{v}$ then $T_\omega(\mathbf{u}) > T_\omega(\mathbf{v})$. Moreover, for $i = 1, \ldots, M$,*
$$T_\omega(\mathbf{u})_i \gg T_\omega(\mathbf{v})_i \quad \text{or} \quad T_\omega(\mathbf{u})_i = T_\omega(\mathbf{v})_i\,.$$
2. *If $\mathbf{u}, \mathbf{v} \in C(\bar{\Omega})^M$ and $\mathbf{u} \gg \mathbf{v}$ then $T_\omega(\mathbf{u}) \gg T_\omega(\mathbf{v})$.*

Proof. 1) For $\varphi \in \mathcal{D}^+(\Omega)$ we have that

$$\begin{aligned}
&\int_\Omega \left(T_\omega(\mathbf{v})_1 - T_\omega(\mathbf{u})_1\right)(-\Delta + \omega)\,\varphi\,dx \\
&= \int_\Omega \left(F_1(\mathbf{v}) + \omega v_1 - F_1(\mathbf{u}) - \omega u_1\right)\varphi\,dx \\
&= \int_\Omega \left(F_1(\mathbf{v}) + \omega v_1 - \left(F_1(u_1, v_2, \ldots, v_M) + \omega u_1\right)\right)\varphi\,dx \\
&\quad + \int_\Omega \left(F_1(u_1, v_2, \ldots, v_M) - F_1(u_1, u_2, v_3, \ldots, v_M)\right)\varphi\,dx \\
&\quad \vdots \\
&\quad + \int_\Omega \left(F_1(u_1, u_2, \ldots, u_{M-1,} v_M) - F_1(\mathbf{u})\right)\varphi\,dx \\
&\le 0.
\end{aligned}$$

By the strong maximum principle, [9, Theorem 9.6], we have either that $T_\omega(\mathbf{u})_1 \gg T_\omega(\mathbf{v})_1$ or that $T_\omega(\mathbf{u})_1 = T_\omega(\mathbf{v})_1$.

For some $x \in \Omega$ and $i \in \{1, \ldots, M\}$ it holds that $u_i(x) > v_i(x)$. From (8) it follows that there exists $\varphi_0 \in \mathcal{D}^+(\Omega)$ such that

$$\int_\Omega \left(F_i(\mathbf{v}) + \omega v_1 - F_i(\mathbf{u}) - \omega u_i\right)\varphi_0\,dx < 0$$

and consequently

$$\int_\Omega \left(T_\omega(\mathbf{v})_i - T_\omega(\mathbf{u})_i\right)(-\Delta + \omega)\,\varphi_0\,dx < 0. \tag{9}$$

Hence $T_\omega(\mathbf{u})_i \gg T_\omega(\mathbf{v})_i$ and $T_\omega(\mathbf{u}) > T_\omega(\mathbf{v})$.

2) Let $i \in \{1, \ldots, M\}$ be fixed. By the first part we have two possibilities: $T_\omega(\mathbf{u})_i \gg T_\omega(\mathbf{v})_i$ or $T_\omega(\mathbf{u})_i = T_\omega(\mathbf{v})_i$. Since $u_i(x) > v_i(x)$ for every $x \in \Omega$ there exists functions $\varphi \in \mathcal{D}^+(\Omega)$, such that

$$\int_\Omega (F_i(\mathbf{v}) + \omega v_1 - F_i(\mathbf{u}) - \omega u_i)\, \varphi \, dx > 0.$$

It follows that $T_\omega(\mathbf{u})_i \gg T_\omega(\mathbf{v})_i$ must hold. Hence $T_\omega(\mathbf{u}) \gg T_\omega(\mathbf{v})$. □

Lemma 5 *Suppose that* $\mathbf{u}$ *is a supersolution for (5) and that* $\omega > 0$ *is such that (8) holds. With* T_ω *as defined in (7) we have that*

1. $T_\omega(\mathbf{u}) = \mathbf{u}$ *and* $\mathbf{u}$ *is a solution to (5) or*

2. $T_\omega(\mathbf{u}) < \mathbf{u}$ *and if* $T_\omega(\mathbf{u})_i < u_i$ *then* $T_\omega(\mathbf{u})_i \ll u_i$*. Moreover* $T_\omega(\mathbf{u})$ *is again a supersolution for (5).*

Similar results hold for subsolutions.

We remark that if the Jacobian matrix in $\mathbf{u}$, $F'(\mathbf{u}) = [\partial F_i(\mathbf{u}) / \partial x_j]$ is fully coupled, see [16] and [14], then a strong maximum principle holds, i.e. in the lemma above either $T_\omega(\mathbf{u}) \gg \mathbf{u}$ or $T_\omega(\mathbf{u}) = \mathbf{u}$.

Proof. 1) A standard bootstrapping argument shows if $T_\omega(\mathbf{u}) = \mathbf{u}$ with $\mathbf{u} \in C(\bar{\Omega})^M$ then $\mathbf{u} \in (C^{2,\alpha}(\bar{\Omega}))^M$. Also

$$(-\Delta + \omega)\, u_i = F(\mathbf{u}) + \omega u_i \qquad \text{in } \Omega$$

and $u_i = 0$ on $\partial\Omega$ so that $\mathbf{u}$ is a solution to (5).

2) Using the definitions of T_ω and a supersolution, we have for $\varphi \in \mathcal{D}^+(\Omega)$ that

$$\begin{aligned} &\int_\Omega (T_\omega(\mathbf{u})_1 - u_1)(-\Delta + \omega)\, \varphi \, dx \\ &= \int_\Omega (F_1(\mathbf{u}) + \omega u_1)\, \varphi \, dx - \int_\Omega u_1 (-\Delta + \omega)\, \varphi \, dx \\ &= \int_\Omega (F_1(\mathbf{u}))\, \varphi \, dx - \int_\Omega u_1 (-\Delta\varphi)\, dx \le 0, \end{aligned}$$

and hence $T_\omega(\mathbf{u})_1 \le u_1$. It even holds that $T_\omega(\mathbf{u})_1 \ll u_1$ or that $T_\omega(\mathbf{u})_1 = u_1$. Suppose namely that $T_\omega(\mathbf{u})_1 < u_1$. Choose $\omega' > \omega$. Similar as for ω we find that $T_{\omega'}(\mathbf{u})_1 \le u_1$. Because $T_\omega(\mathbf{u})_1 < u_1$ one has for all $\varphi \in \mathcal{D}^+(\Omega)$ that

$$\begin{aligned}
&\int_\Omega \left(T_\omega(\mathbf{u})_1 - T_{\omega'}(\mathbf{u})_1\right)(-\Delta + \omega')\,\varphi\,dx \\
&\quad= \int_\Omega \left(T_\omega(\mathbf{u})_1\right)(-\Delta + \omega)\,\varphi\,dx - (\omega - \omega')\int_\Omega \left(T_\omega(\mathbf{u})_1\right)\varphi\,dx \\
&\qquad - \int_\Omega \left(F_1(\mathbf{u}) + \omega' u_1\right)\varphi\,dx \\
&\quad= \int_\Omega \left(F_1(\mathbf{u}) + \omega u_1\right)\varphi\,dx - (\omega - \omega')\int_\Omega \left(T_\omega(\mathbf{u})_1\right)\varphi\,dx \\
&\qquad - \int_\Omega \left(F_1(\mathbf{u}) + \omega' u_1\right)\varphi\,dx \\
&\quad= (\omega - \omega')\int_\Omega (u_1 - T_\omega(\mathbf{u})_1)\varphi\,dx \le 0,
\end{aligned}$$

with strict inequality for certain $\varphi \in \mathcal{D}^+(\Omega)$. Since $T_\omega(\mathbf{u})_1 \in W^{2,p}_{\mathrm{loc}} \cap C(\bar{\Omega})$ the strong maximum principle, [9, Theorem 9.6], implies that $T_\omega(\mathbf{u})_1 \ll T_{\omega'}(\mathbf{u})_1 \le u_1$. In the same way for $i = 2, \ldots, M$, it holds that $T_\omega(\mathbf{u})_i \ll u_i$ or $T_\omega(\mathbf{u})_i = u_i$. In particular, $T_\omega(\mathbf{u}) \ne \mathbf{u}$ then $T_\omega(\mathbf{u}) < \mathbf{u}$. If this is the case then for $\varphi \in \mathcal{D}^{+,M}(\Omega)$ it holds that

$$\begin{aligned}
\int_\Omega T_\omega(\mathbf{u}) \cdot (-\Delta\varphi)\;dx &= \int_\Omega \left(F_\omega(\mathbf{u}) - \omega\mathbf{u}\right) \cdot \varphi\,dx \\
&\ge \int_\Omega \left(F_\omega(T_\omega(\mathbf{u})) - \omega T_\omega(\mathbf{u})\right) \cdot \varphi\,dx \\
&= \int_\Omega F(T_\omega(\mathbf{u})) \cdot \varphi\,dx,
\end{aligned}$$

with strict inequality for some φ. Also $T_\omega(\mathbf{u}) = \mathbf{0}$ on $\partial\Omega$ and hence $T_\omega(\mathbf{u})$ is a strict supersolution. □

We will use the following variation of a theorem due to H. Amann, see [2, Theorem 2.6, Corollary 2.7 and the remark after the proof of 2.7] as well as [1, Theorem 14.2].

Theorem 6 (Amann) *Let E be a ordered Banach space of which the positive cone P has a nonempty interior P°. Suppose there exist two pairs $(\bar{y}, \bar{z})$, $(\tilde{y}, \tilde{z}) \in E^2$ with*

$$\bar{y} < \bar{z} < \tilde{z}, \quad \bar{y} < \tilde{y} < \tilde{z}, \quad \text{and} \quad \bar{z} \not\geq \tilde{y}, \tag{10}$$

and a compact increasing map $f : [\bar{y}, \tilde{z}] \to E$ *such that*

$$\bar{y} \leq f(\bar{y}), \quad f(\bar{z}) < \bar{z}, \quad \tilde{y} < f(\tilde{y}), \quad f(\tilde{z}) \leq \tilde{z}. \tag{11}$$

Then f *has a maximal fixed point* $\bar{x} \in [\bar{y}, \bar{z}]$ *and a minimal fixed point* $\tilde{x} \in [\tilde{y}, \tilde{z}]$. *If* $\bar{z} - \bar{x} \in P^\circ$ *and* $\tilde{x} - \tilde{y} \in P^\circ$ *then* f *has a third fixed point* x *such that* $x \not\leq \bar{z}$ *and* $x \not\geq \tilde{y}$. *Moreover* $x_{\min} < x < x_{\max}$ *where* $x_{\min}$ *and* $x_{\max}$ *are respectively the minimal and maximal fixed points of* f *in* $[\bar{y}, \tilde{z}]$.

This theorem will be used in the following setting (see [1]). Let $e \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ be the solution of

$$\begin{cases} -\Delta e &= 1 \qquad \text{in } \Omega, \\ e &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

and let

$$C_e(\bar{\Omega}) := \left\{ u \in C(\bar{\Omega}); |u| \leq \lambda e \text{ for some } \lambda \geq 0 \right\}.$$

Equipped with the norm

$$\|u\|_e := \inf \{\lambda > 0; |u| \leq \lambda e\},$$

$C_e(\bar{\Omega})$ is a ordered Banach space and the positive cone has a nonempty interior. In fact $u \in C_e(\bar{\Omega})$ is in the interior of the positive cone if and only if $u \geq \lambda e$ for some $\lambda > 0$. The following lemma holds.

Lemma 7 *With* ω *chosen as in (8), let* $S : C_e(\bar{\Omega})^M \to C_e(\bar{\Omega})^M$ *be the restriction of* T_ω *to* $C_e(\bar{\Omega})^M$. *Then* S *is continuous, compact and increasing. Moreover, if* $\mathbf{u} \ll \mathbf{v}$ *then* $S\mathbf{v} - S\mathbf{u} \geq \lambda \mathbf{e}$ *for some* $\lambda > 0$, *where* $\mathbf{e} := (e, \ldots, e)$.

Proof. The operator K_ω defined earlier maps $C(\bar{\Omega})$ compactly into $C_e(\bar{\Omega})$. To see this we note that K_ω maps $C(\bar{\Omega})$ compactly into $C_0^1(\bar{\Omega}) := \left\{ u \in C^1(\bar{\Omega}); u_{|\partial\Omega} = 0 \right\}$ and that $C_0^1(\bar{\Omega})$ is continuously imbedded in $C_e(\bar{\Omega})$. Consequently the mapping $\mathbf{u} \mapsto (K_\omega F_{\omega,1}(\mathbf{u}), \ldots, K_\omega F_{\omega,1}(\mathbf{u}))$ of $C(\bar{\Omega})^M$ into $C_e(\bar{\Omega})^M$ is compact and hence S is compact.

If $\mathbf{u} \ll \mathbf{v}$ then for every $i = 1, \ldots, M$ and $\varphi \in \mathcal{D}^+(\Omega)$

$$\int_\Omega ((S\mathbf{v})_i - (S\mathbf{u})_i)(-\Delta + \omega) \varphi \, dx \geq 0$$

with strict inequality for some φ. By remark after Definition 3 the result follows. □

We can now reformulate Theorem 6 as a result about the existence and multiplicity of solutions to (5).

Proposition 8 *Suppose that* $\bar{\mathbf{v}}, \tilde{\mathbf{v}}$ *are two subsolutions of (5), with* $\tilde{\mathbf{v}}$ *strict and that* $\bar{\mathbf{w}}, \tilde{\mathbf{w}}$ *are two supersolutions of (5) with* $\bar{\mathbf{w}}$ *strict. If*

$$\bar{\mathbf{v}} < \bar{\mathbf{w}} < \tilde{\mathbf{w}}, \qquad \bar{\mathbf{v}} < \tilde{\mathbf{v}} < \tilde{\mathbf{w}} \qquad \textit{and} \qquad \bar{\mathbf{w}} \not\geq \tilde{\mathbf{v}},$$

then (5) has two solutions $\bar{\mathbf{u}}, \tilde{\mathbf{u}}$ *with*

$$\bar{\mathbf{v}} \leq \bar{\mathbf{u}} < \bar{\mathbf{w}} \quad \textit{and} \quad \tilde{\mathbf{v}} < \tilde{\mathbf{u}} \leq \tilde{\mathbf{w}}.$$

If $\bar{\mathbf{u}} \ll \bar{\mathbf{w}}$ *and* $\tilde{\mathbf{u}} \gg \tilde{\mathbf{v}}$ *then there exists a third solution* $\mathbf{u}$ *such that*

$$\mathbf{v}_{\min} < \mathbf{u} < \mathbf{w}_{\max}, \qquad \mathbf{u} \not\leq \bar{\mathbf{w}} \qquad \textit{and} \qquad \mathbf{u} \not\geq \tilde{\mathbf{v}}$$

where $\mathbf{v}_{\min}$ *and* $\mathbf{w}_{\max}$ *are respectively the minimal and maximal solutions in* $[\bar{\mathbf{v}}, \tilde{\mathbf{w}}]$.

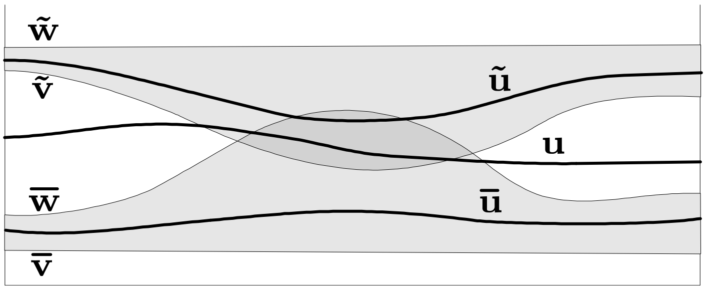

Figure 2

Proof. Since the sub- and supersolutions need not be in $C_e(\bar{\Omega})$ we cannot apply Theorem 6 directly. Let ω be chosen as in (8) and define T_ω as in (7). Let

$$\bar{\mathbf{y}} := T_\omega \bar{\mathbf{v}}, \qquad \bar{\mathbf{z}} := T_\omega \bar{\mathbf{w}}, \qquad \tilde{\mathbf{y}} := T_\omega \tilde{\mathbf{v}}, \qquad \tilde{\mathbf{z}} := T_\omega \tilde{\mathbf{w}}.$$

By Lemmas 4 and 5 it holds that

$$(\bar{\mathbf{v}} \leq)\ \bar{\mathbf{y}} < \bar{\mathbf{z}} < \tilde{\mathbf{z}}\ \ (\leq \tilde{\mathbf{w}}), \quad \text{and} \quad (\bar{\mathbf{v}} \leq)\ \bar{\mathbf{y}} < \tilde{\mathbf{y}} < \tilde{\mathbf{z}}\ \ (\leq \tilde{\mathbf{w}}),$$

with $\bar{\mathbf{y}}, \tilde{\mathbf{y}}$ again subsolutions and $\bar{\mathbf{z}}, \tilde{\mathbf{z}}$ again supersolutions. Because $\bar{\mathbf{w}}$ is a supersolution and $\tilde{\mathbf{v}}$ is a subsolution we have that $\tilde{\mathbf{y}} \geq \tilde{\mathbf{v}}$ and $\bar{\mathbf{z}} \leq \bar{\mathbf{w}}$. Hence $\bar{\mathbf{z}} \not\geq \tilde{\mathbf{y}}$ because $\bar{\mathbf{z}} \geq \tilde{\mathbf{y}}$ implies $\bar{\mathbf{w}} \geq \tilde{\mathbf{v}}$, a contradiction.

The restriction $S : C_e(\bar{\Omega})^M \to C_e(\bar{\Omega})^M$ of T_ω to $C_e(\bar{\Omega})^M$ is compact and increasing by Lemma 7. Also, by Lemma 5,

$$\tilde{\mathbf{y}} \leq S\tilde{\mathbf{y}}, \qquad S\bar{\mathbf{z}} < \bar{\mathbf{z}}, \qquad \bar{\mathbf{y}} < S\bar{\mathbf{y}}, \qquad S(\tilde{\mathbf{z}}) \leq \tilde{\mathbf{z}}.$$

By the theorem, S has fixed two points, say $\bar{\mathbf{u}}$,$\tilde{\mathbf{u}} \in C_e(\bar{\Omega})^M$ such that $\bar{\mathbf{y}} \leq \bar{\mathbf{u}} < \bar{\mathbf{z}}$ and $\tilde{\mathbf{y}} < \tilde{\mathbf{u}} \leq \tilde{\mathbf{z}}$.

If $\bar{\mathbf{u}} \ll \bar{\mathbf{w}}$ and $\tilde{\mathbf{u}} \gg \tilde{\mathbf{v}}$ we have by Lemma 7 that $S\bar{\mathbf{w}} - S\bar{\mathbf{u}} = \bar{\mathbf{z}} - \bar{\mathbf{u}}$ and $S\tilde{\mathbf{u}} - S\tilde{\mathbf{v}} = \tilde{\mathbf{u}} - \tilde{\mathbf{y}}$ are in the interior of the positive cone of $C_e(\bar{\Omega})^M$. The existence of a third fixed point, say $\mathbf{u} \in C_e(\bar{\Omega})^M$ with $\mathbf{u} \not\leq \bar{\mathbf{z}}$ and $\mathbf{u} \not\geq \tilde{\mathbf{y}}$ follows from the theorem. Moreover, $\mathbf{u} \not\leq \bar{\mathbf{w}}$ for if $\mathbf{u} \leq \bar{\mathbf{w}}$ then $\mathbf{u} = S\mathbf{u} \leq S\bar{\mathbf{w}} = \bar{\mathbf{z}}$, a contradiction. In the same way $\mathbf{u} \not\geq \tilde{\mathbf{v}}$. Also by the theorem $\mathbf{v}_{\min} < \mathbf{u} < \mathbf{w}_{\max}$. Since fixed points of S are solutions to (5), the corollary follows. □

Finally we state the following proposition which was proved by Troy, [17, Theorem 1] about positive solutions to a quasimonotone system. The results of Troy generalize the results of Gidas, Ni and Nirenberg, [8], for the scalar case.

Proposition 9 (Troy) *Suppose that (5) is quasimonotone and that* $\Omega = B_R$*. If* $\mathbf{u} \gg \mathbf{0}$ *is a solution to this system with* $u_i \in C^2\left(\bar{B}_R\right)$, $i = 1, \ldots, M$, *then* u_i *is radially symmetric and* $\partial u_i / \partial r < 0$ *on* $(0, R)$.

4 Proof of the main result

Our aim is to construct a solution to (1) by using solutions to (3) and letting $R \to \infty$. To obtain the solutions on B_R we shall construct a quasimonotone system of which the solutions in an appropriate range can be used to obtain solutions to (3). If $z = u - \beta v$, then

$$\begin{aligned}
-\Delta z &= -\Delta u + \beta \Delta v \\
&= f(u) - v - \delta\beta u + \delta\beta\gamma v \\
&= f(u) + (1-a)u + (a-1)(z+\beta v) - v + \delta\beta\gamma v - \delta\beta(z+\beta v) \\
&= f(u) + (1-a)u + \left(-\delta\beta^2 + (\delta\gamma + a - 1)\beta - 1\right)v + (a - 1 - \delta\beta)z.
\end{aligned}$$

Taking $\beta := \frac{1}{2\delta}(\delta\gamma + a - 1) + \frac{1}{2\delta}\sqrt{(\delta\gamma + a - 1)^2 - 4\delta}$, as in condition 2, one has that

$$-\Delta z = f(u) + (1-a)u + (a - 1 - \delta\beta)z.$$

By the following modification of f we obtain a quasimonotone system. We define f^* by

$$f^*(u) = \begin{cases} -au & \text{if } u < 0; \\ f(u) & \text{if } 0 \le u \le 1; \\ (a-1)\,u & \text{if } u \ge 1. \end{cases}$$

One has that $f^* \in C^1(\mathbb{R})$ with $\frac{d}{du} f^*(u) \ge (a-1)$. The system

$$\begin{cases} -\Delta u &= f^*(u) - \frac{1}{\beta}u + \frac{1}{\beta}z & \text{in } B_R, \\ -\Delta z &= f^*(u) + (1-a)\,u + (a-1-\delta\beta)\,z & \text{in } B_R, \\ u &= z = 0 & \text{on } \partial B_R. \end{cases} \tag{12}$$

is hence quasimonotone. By the choice of β it follows that if (u,z) is a solution to (12) and $0 \le u \le 1$ then $(u, \frac{1}{\beta}(u-z))$ is a solution to (3). We now use the results of the previous section to prove the following proposition.

Proposition 10 *Suppose that condition 1 and condition 2 hold.*

1. *There exists $R_0 > 0$ such that for every $R \ge R_0$ system (3) possesses two nontrivial solutions (u_R, v_R) and $(\tilde{u}_R, \tilde{v}_R)$ in $C\left(\bar{B}_R\right) \cap C^2(B_R)$.*
2. *For the solutions it holds that $(0,0) < (u_R, v_R) < (\tilde{u}_R, \tilde{v}_R) < (\sigma_\gamma, \frac{1}{\gamma}\sigma_\gamma)$.*
3. *The solutions are radially symmetric decreasing functions.*
4. *For every $R \ge R_0$ it holds that*

$$u_R(x) < \sigma_\beta \qquad \text{for all} \qquad |x| \ge R_0, \tag{13}$$

and

$$u_R(0) \ge \theta_\gamma. \tag{14}$$

Proof. We will first prove the proposition for a fixed $R_0 > 0$ large enough, and then show how the proposition follows for $R > R_0$.

In order to apply Corollary 8 we need appropriate sub- and supersolutions. One has that

$$\bar{\mathbf{v}} = (0,0)\,, \quad \bar{\mathbf{w}} = \left(\theta_\gamma, (1-\frac{\beta}{\gamma})\theta_\gamma\right), \text{and} \quad \tilde{\mathbf{w}} = \left(\sigma_\gamma, (1-\frac{\beta}{\gamma})\sigma_\gamma\right)$$

are respectively, a subsolution and two strict supersolutions with $\bar{\mathbf{v}} < \bar{\mathbf{w}} < \tilde{\mathbf{w}}$. We still need a strict subsolution $\tilde{\mathbf{v}}$ such that $\bar{\mathbf{v}} < \tilde{\mathbf{v}} < \tilde{\mathbf{w}}$ and $\tilde{\mathbf{v}} \not\leq \bar{\mathbf{w}}$. Since

$$\int_0^{\sigma_\beta} \left(f(s) - \frac{s}{\beta} \right) ds > 0$$

there exists $R_0 > 0$ such that the problem

$$\begin{cases} -\Delta\varphi &= f(\varphi) - \frac{1}{\beta}\varphi & \text{in } B_{R_0}, \\ \varphi &= 0 & \text{on } \partial B_{R_0}, \end{cases}$$

possesses a nontrivial positive solution with

$$\theta_\beta < \max_{x \in B_{R_0}} \varphi(x) < \sigma_\beta,$$

see [3] and [5]. If we set

$$\tilde{\mathbf{v}} = (\varphi, 0)$$

then $\tilde{\mathbf{v}}$ is a suitable subsolution. By Corollary 8 there exist two solutions $\bar{\mathbf{u}}_{R_0}$ and $\tilde{\mathbf{u}}_{R_0}$ with $\mathbf{0} \leq \bar{\mathbf{u}}_{R_0} < \bar{\mathbf{w}}$ and $(\varphi, 0) < \tilde{\mathbf{u}}_{R_0} \leq \tilde{\mathbf{v}}$. We will omit the subscripts R_0 while working in B_{R_0}.

We show that the condition for a third solution is satisfied. In B_{R_0}, with $\bar{\mathbf{u}} = (\bar{u}, \bar{z})$, we have that

$$\begin{aligned} \left(-\Delta + \frac{1}{\beta}\right)(\bar{u} - \theta_\gamma) &= f(\bar{u}) - \frac{1}{\beta}\bar{u} + \frac{1}{\beta}\bar{w} + \frac{1}{\beta}(\bar{u} - \theta_\gamma) \\ &\leq f(\bar{u}) + \frac{1}{\beta}(1 - \frac{\beta}{\gamma})\theta_\gamma - \frac{1}{\beta}\theta_\gamma \\ &= f(\bar{u}) - \frac{1}{\gamma}\theta_\gamma \leq 0. \end{aligned}$$

By the strong maximum principle, $\bar{u}(x) < \theta_\gamma$ for all $x \in B_{R_0}$. Also, using the definition of β, we have that

$$\begin{aligned} &(-\Delta + (1 + \delta\beta - a))\left(\bar{z} - (1 - \frac{\beta}{\gamma})\theta_\gamma\right) \\ &\quad = f(\bar{u}) + (1 - a)\bar{u} - \left(1 - a + \frac{1}{\gamma}\right)\theta_\gamma \\ &\quad = f(\bar{u}) - \frac{1}{\gamma}\theta_\gamma + (1 - a)(\bar{u} - \theta_\gamma) \leq 0. \end{aligned}$$

Again by the strong maximum principle, $\bar{z}(x) < (1-\frac{\beta}{\gamma})\theta_\gamma$ for all $x \in B_{R_0}$. Hence $\bar{\mathbf{u}} \ll \bar{\mathbf{w}}$. That $\tilde{\mathbf{v}} \ll \tilde{\mathbf{u}} = (\tilde{u}, \tilde{z})$ follows in the same way by considering $(-\Delta + \omega)(\tilde{u} - \varphi)$ with $\omega > 0$ large enough and $(-\Delta + (1 + \delta\beta - a))\tilde{z}$.

Without loss of generality assume that $\bar{\mathbf{u}} = \mathbf{0}$, the minimal solution in $[\mathbf{0}, \tilde{\mathbf{w}}]$ and that $\tilde{\mathbf{u}}$ is the maximal solution in $[\mathbf{0}, \tilde{\mathbf{w}}]$.

The existence of a third solutions $\mathbf{0} < \mathbf{u} = (u, z) < \tilde{\mathbf{u}}$ such that

$$\mathbf{u} \not\leq \left(\theta_\gamma, (1-\frac{\beta}{\gamma})\theta_\gamma\right) \qquad \text{and} \qquad \mathbf{u} \not\geq (\varphi, 0)$$

follows from Corollary 8. In particular, for some $\xi_{R_0} \in B_{R_0}$

$$u(\xi_{R_0}) > \theta_\gamma, \tag{15}$$

because if $u \leq \theta_\gamma$ then,as above, $z \leq (1-\frac{\beta}{\gamma})\theta_\gamma$ which contradicts the fact that $\mathbf{u} \not\leq \left(\theta_\gamma, (1-\frac{\beta}{\gamma})\theta_\gamma\right)$. We also have the existence of $\zeta_{R_0} \in B_{R_0}$ such that $\mathbf{u}(\zeta_{R_0}) < (\varphi(\zeta_{R_0}), 0)$ and hence

$$u(\zeta_{R_0}) < \sigma_\beta. \tag{16}$$

With $v := \frac{1}{\beta}(u - z)$, we have that (u, v) is a solution to (3) with $R = R_0$. In particular, the second equation of (3) implies that $v \geq 0$ and by the maximum principle and the fact that $u < \sigma_\gamma$ we also have that $v < \sigma_\gamma/\gamma$. An application of the strong maximum principle shows that $\mathbf{u} \gg \mathbf{0}$. By Proposition 9, u is radially symmetric and decreasing. Because u decreases, (13) and (14) follows from (15) and (16). We also have that v is radially symmetric and decreasing because $(-\Delta + \delta\gamma)\, v = u$ in B_{R_0}, $v = 0$ on ∂B_{R_0}.

On the balls B_R with $R > R_0$ the proof is similar. As sub- and supersolutions we choose

$$\check{\mathbf{w}} = (0, 0)\,, \;\; \check{\mathbf{v}} = (\varphi_R, 0)\,, \;\; \hat{\mathbf{w}} = \left(\theta_\gamma, (1-\frac{\beta}{\gamma})\theta_\gamma\right) \;\text{ and }\; \hat{\mathbf{v}} = \left(\sigma_\gamma, (1-\frac{\beta}{\gamma})\sigma_\gamma\right).$$

Here φ_R the trivial extension to B_R of the function φ used in the first part. Note that it is this choice of subsolution from which we get that $u_R(\zeta_R) < \sigma_\beta$ with $\zeta_R \in B_{R_0}$. From this and the fact that u_R decreases, (13) follows. □

We now prove the main result.

Proof. (Theorem 1) We will use the functions $\{u_R\,;\, R \geq R_0\}$ and $\{v_R\,;\, R \geq R_0\}$ obtained in the previous theorem, to construct a solution to system (1). We extend these functions trivially to functions on $\mathbb{R}^N$.

The proof is given in eight steps. The idea is as follows. We show $\{u_R\}$ and $\{v_R\}$ are precompact in $C^2_{\text{loc}}(\mathbb{R}^N)$. Using this we obtain a solution on $\mathbb{R}^N$. Properties (13) and (14) will then be used to show that the solution obtained is nontrivial and goes to zero at infinity.

Step 1. Let $R_1 > R_0$. The functions $\{u_R\,;\, R \geq 2R_1\}$ and $\{v_R\,;\, R \geq 2R_1\}$ satisfy

$$\begin{cases} -\Delta u_R &= f(u_R) - v_R & \text{on } B_{2R_1}, \\ -\Delta v_R &= \delta(u_R - \gamma v_R) & \text{on } B_{2R_1}, \end{cases}$$

and from Proposition 10 we have that

$$\|u_R\|_{L^\infty(B_{2R_1})} \leq \theta_\gamma, \quad \|v_R\|_{L^\infty(B_{2R_1})} \leq (1/\gamma)\,\theta_\gamma,$$

and also

$$\|f(u_R)\|_{L^\infty(B_{2R_1})} \leq \theta_f := \sup_{0 \leq x \leq 1} |f(x)|\,.$$

Using interior elliptic estimates, [9, Theorem 9.11], Schauder interior estimates, [9, Problem 6.1], and the fact that f is locally Lipschitz, we that find $\{u_R\,;\, R \geq 2R_1\}$ and $\{v_R\,;\, R \geq 2R_1\}$ are bounded in $C^{2,1}(\bar{B}_{R_1})$ and hence precompact in $C^2(\bar{B}_{R_1})$. Then there exist sequences $\{u_{R_1^{(n)}}\}$ and $\{v_{R_1^{(n)}}\}$ ($R_1 < R_1^{(n)} \nearrow \infty$ if $n \to \infty$) which converge in $C^2(\bar{B}_{R_1})$. We set for $x \in \bar{B}_{R_1}$

$$u_1(x) = \lim_{n\to\infty} u_{R_1^{(n)}}(x) \qquad \text{and} \qquad v_1(x) = \lim_{n\to\infty} v_{R_1^{(n)}}(x)\,.$$

On B_{R_1} the functions u_1, v_1 are solutions of the equations.

Step 2. Let $R_2 := R_1^{(1)}$ and repeat step 1 to obtain that $\{u_{R_1^{(n)}}\}$ and $\{v_{R_1^{(n)}}\}$ are bounded sequences in $C^{2,1}(B_{R_2})$. Again we can extract subsequences $\{u_{R_2^{(n)}}\}$ and $\{v_{R_2^{(n)}}\}$ from $\{u_{R_1^{(n)}}\}$ and $\{v_{R_1^{(n)}}\}$ respectively such that they converge in $C^2(\bar{B}_{R_2})$. We extend the functions u_1 and v_1 to B_{R_2} by defining for $x \in B_{R_2}$

$$u_2(x) = \lim_{n\to\infty} u_{R_2^{(n)}}(x) \qquad \text{and} \qquad v_1(x) = \lim_{n\to\infty} v_{R_1^{(n)}}(x)\,,$$

These functions satisfy the equations on B_{R_2}.

Step 3. By repeating this process we obtain for every $k \in \mathbb{N}$ two sequences $\{u_{R_k^{(n)}}\}$ and $\{v_{R_k^{(n)}}\}$ converging in $C^2(\bar{B}_{R_k})$ which are subsequences of $\{u_{R_{k-1}^{(n)}}\}$ and $\{v_{R_{k-1}^{(n)}}\}$ and $u_k = \lim_{n\to\infty} u_{R_k^{(n)}}$, $v_k = \lim_{n\to\infty} v_{R_k^{(n)}}$ satisfy the equations on B_{R_k}.

Step 4. For the diagonal sequences $\{u_{R_m^{(m)}}\}$ and $\{v_{R_m^{(m)}}\}$ one has for every $x \in B_{R_k}$ that

$$u_k(x) = \lim_{m\to\infty} u_{R_m^{(m)}}(x) \qquad \text{and} \qquad v_k(x) = \lim_{m\to\infty} v_{R_m^{(m)}}(x).$$

Hence, if $u = \lim_{m\to\infty} u_{R_m^{(m)}}$ and $v = \lim_{m\to\infty} v_{R_m^{(m)}}$ then

$$\begin{cases} -\Delta u &= f(u) - v & \text{on } \mathbb{R}^N, \\ -\Delta v &= \delta(u - \gamma v) & \text{on } \mathbb{R}^N. \end{cases}$$

Step 5. By Proposition 10

$$u_R(0) = \max_{x\in B_R} u_R(x) \geq \left(1 - \frac{\beta}{\gamma}\right)\theta_\gamma,$$

and hence

$$u(0) = \max_{x\in\mathbb{R}^N} u(x) \geq \left(1 - \frac{\beta}{\gamma}\right)\theta_\gamma > 0.$$

Consequently $u, v \neq 0$

Step 6. It remains to show that $u(x), v(x) \to 0$ if $|x| \to \infty$. Since all the functions u_R and v_R are radially symmetric we will consider u_R, v_R, u, v as functions of one variable. In particular we have that $u'(r) \leq 0$, $v'(r) \leq 0$ for all $r > 0$ and $u'(0) = v'(0) = 0$. Let

$$l_u := \lim_{r\to\infty} u(r) = \inf_{r>0} u(r) \qquad \text{and} \qquad l_v := \lim_{r\to\infty} v(r) = \inf_{r>0} v(r). \tag{17}$$

In step 7 we shall show that

$$l_u \in \{0, \theta_\gamma, \sigma_\gamma\} \quad \text{and} \quad l_v = l_u/\gamma. \tag{18}$$

Then, by Proposition 10,

$$l_u \leq u(R_0) \leq \sigma_\beta < \sigma_\gamma$$

so that $l_u \neq \sigma_\gamma$. To exclude the possibility $l_u = \theta_\gamma$ we shall show in step 8 that

$$\int_0^{l_u} \left(f(t) - \frac{t}{\gamma} \right) dt = F(l_u) - l_u^2/(2\gamma) \geq 0, \tag{19}$$

which cannot hold for $l_u = \theta_\gamma$. Then the only remaining possibility is that $l_u = l_v = 0$.

Step 7. We prove (18). Because of the radial symmetry we have that

$$\begin{cases} -u'' - \frac{N-1}{r}u' &= f(u) - v & r > 0, \\ -v'' - \frac{N-1}{r}v' &= \delta u - \delta\gamma v & r > 0, \\ u'(0) &= v'(0) = 0. \end{cases} \tag{20}$$

Let $u_0 := u(0)$ and $v_0 := v(0)$. Multiplying the first equation with u' and the second with v' and integrating one finds that

$$\frac{1}{2}u'(R)^2 + (N-1)\int_0^R \frac{(u')^2}{r}\,dr = F(u_0) - F(u(R)) + \int_0^R u'v\,dr,$$

and

$$\begin{aligned} &\frac{1}{2}v'(R)^2 + (N-1)\int_0^R \frac{(v')^2}{r}\,dr \\ &= -\delta(u(R)v(R) - u_0v_0) + \delta\int_0^R u'v\,dr + (\delta\gamma/2)\left(v(R)^2 - v_0^2\right) \end{aligned}$$

Adding we find that

$$\begin{aligned} &\frac{1}{2}\left(u'(R)^2 + \delta^{-1}v'(R)^2\right) + \\ &\quad + (N-1)\int_0^R \frac{(u')^2 + \delta^{-1}(v')^2}{r}\,dr - 2\int_0^R uv\,dr \\ &= F(u_0) - F(u(R)) - (u(R)v(R) - u_0v_0) + \frac{\gamma}{2}\left(v(R)^2 - v_0^2\right), \end{aligned} \tag{21}$$

and subtracting that

$$\begin{aligned} &\frac{1}{2}\left(u'(R)^2 - \delta^{-1}v'(R)^2\right) + (N-1)\int_0^R \frac{(u')^2 - \delta^{-1}(v')^2}{r}\,dr \\ &= F(u_0) - F(u(R)) - u_0v_0 + u(R)v(R) - (\gamma/2)\left(v(R)^2 - v_0^2\right). \end{aligned} \tag{22}$$

Because $u'(R), v'(R) \leq 0$ and $u(R), v(R)$ stay bounded as $R \to \infty$ we have from (21) that

$$u'(R) \to 0 \quad \text{and} \quad v'(R) \to 0 \quad \text{as} \quad R \to \infty.$$

Also, we see from (20) that

$$u''(R) \to f(l_u) - l_v \quad \text{and} \quad -v''(R) \to \delta l_u - \delta\gamma l_v \quad \text{if } R \to \infty,$$

so that $f(l_u) - l_v = 0$ and $\delta l_u - \delta\gamma l_v = 0$ and hence (18) holds.

Step 8. Finally we prove (19). We note that

$$u_R'(r)^2 - \delta^{-1} v_R'(r)^2 \geq 0. \tag{23}$$

This follows from the fact that $\sqrt{\delta}\beta - 1 \geq 0$. Indeed, with z_R as in the proof of Proposition 10,

$$\begin{aligned} u_R'(r) - \delta^{-1/2} v_R'(r) &= u_R'(r) - (\beta\sqrt{\delta})^{-1} u_R'(r) + (\beta\sqrt{\delta})^{-1} z_R'(r) \\ &= (\beta\sqrt{\delta})^{-1} \left(\sqrt{\delta}\beta - 1\right) u_R'(r) + (\beta\sqrt{\delta})^{-1} z_R'(r) \\ &\leq 0 \end{aligned}$$

and hence

$$u_R'(r)^2 - \delta^{-1} v_R'(r)^2 = \left(u_R'(r) - \delta^{-1/2} v_R'(r)\right)\left(u_R'(r) + \delta^{-1/2} v_R'(r)\right) \geq 0.$$

$>$From (22) we see by letting $R \to \infty$ that

$$\begin{aligned} &(N-1) \int_0^\infty \frac{u'(r)^2 - \delta^{-1} v'(r)^2}{r}\, dr \\ &= F(u_0) - F(l_u) - u_0 v_0 + l_u^2/(2\gamma) + (\gamma/2) v_0^2. \end{aligned} \tag{24}$$

On the other hand, for every solution (u_R, v_R) it holds that

$$\begin{aligned} &\frac{1}{2}\left(u_R'(R)^2 - \delta^{-1} v_R'(R)^2\right) + (N-1) \int_0^R \frac{u_R'(r)^2 - \delta^{-1} v_R'(r)^2}{r}\, dr \\ &= F(u_R(0)) - u_R(0)\, v_R(0) + (\gamma/2) v_R(0)^2. \end{aligned}$$

Hence, for a fixed $K > 0$ and all $R \geq K$ it holds that

$$\begin{aligned} &(N-1) \int_0^K \frac{u_R'(r)^2 - \delta^{-1} v_R'(r)^2}{r}\, dr \\ &\leq F(u_R(0)) - u_R(0)\, v_R(0) + (\gamma/2) v_R(0)^2 \end{aligned}$$

so that

$$(N-1)\int_0^K \frac{u'(r)^2-\delta^{-1}v'(r)^2}{r}\,dr \leq F(u_0)-u_0v_0+(\gamma/2)v_0^2.$$

Letting $K\to\infty$ we find that

$$(N-1)\int_0^\infty \frac{u'(r)^2-\frac{1}{\delta}v'(r)^2}{r}\,dr \leq F(u_0)-u_0v_0+(\gamma/2)v_0^2. \tag{25}$$

>From (24) and (25) we get that

$$F(u_0)-F(l_u)-u_0v_0+l_u^2/(2\gamma)+(\gamma/2)v_0^2 \leq F(u_0)-u_0v_0+(\gamma/2)v_0^2,$$

which is precisely (19). □

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