Positivity results for a nonlocal elliptic equation

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1 Introduction

Elliptic and parabolic differential equations which include an integral term of the form

$$\int_\Omega g(x, y, u(y))dy,$$

where Ω is the bounded spatial domain and $u(y) = u(y, t)$ in the parabolic case, have appeared in applications for quite a long time – see, for instance, [14] and the references in [7].

However, it has only been recently that more attention has been paid to these equations, and that it has become clear that the behaviour one can expect from such problems is in fact more complicated than that of standard elliptic and parabolic equations. In particular, it is known that, for the time–dependent problem, ω–limit sets can now have an arbitrarily high dimension, and that even in the case where there is no explicit dependence on the space variable it is possible to have stable periodic orbits [6, 9, 17]. Still in this last case, the type of stationary solutions that may now be stable is more varied than in the local case, and it has been shown that these will now include sign–changing and nonconstant solutions in the Dirichlet and Neumann cases, respectively [8, 10].

This more complex behaviour is due to the fact that, because of the introduction of the integral term, these equations will no longer satisfy a form of maximum or comparison principle in general, and, in the case of the time–dependent problem, there is no longer a general Lyapunov functional.

It has also been shown that, provided the nonlocal term satisfies some sort of monotonicity condition, then the asymptotic behaviour of solutions of the time dependent problem is in some sense similar to that of the standard parabolic problem, as again a maximum principle can be applied [10].

The purpose of this paper is to give conditions which ensure that the monotonicity is preserved. This monotonicity however does not follow immediately from standard maximum principles but uses estimates of the kernels involved. Such type of estimates are related with the Green function estimates of Ancona ([1]), Hueber and Sieveking ([13]), and the 3G-estimates of Zhao ([19]) and Cranston, Fabes and Zhao ([4]).

We consider a linear inhomogeneous nonlocal elliptic equation of the form

$$\begin{cases} \mathcal{L}u - \varepsilon\mathcal{K}u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha > 0$ and where ε is a real parameter. Here, the operator $\mathcal{L}$ is the uniformly elliptic second order differential operator

$$\mathcal{L} = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

with

$$\begin{gathered} c_1 |\xi|^2 \geq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq c_1^{-1} |\xi|^2 \\ a_{ij}, b_i, c \in C^{0,\alpha}(\overline{\Omega}) \end{gathered} \tag{2}$$

for some $\alpha > 0$ and a_{ij}, b_i and c are such that $\mathcal{L}$ has a positive principal eigenvalue. The class of operators $\mathcal{K}$ that we consider contains integral operators of the form

$$(\mathcal{K}u)(x) = \int_{\Omega} K(x,y) u(y) \, dy, \tag{3}$$

which includes the case of a separable kernel $K(x,y) = f(x) g(y)$ considered by Allegretto and Barabanova ([2]).

In the absence of the integral operator $\mathcal{K}$, it is well–known that if f is a nonnegative function in $L^p(\Omega)$, not identically zero and $p > \frac{1}{2}n$, then there is one and only one solution in $C_0(\overline{\Omega})$ to equation (1) ([11], Theorem 8.30), which is strictly positive in Ω. Moreover, if $p > n$ then ([11], Theorem 9.15) $u \in W^{2,p}(\Omega)$ which is compactly imbedded in $C^1(\overline{\Omega})$. It is easy to see through an example that positivity will no longer hold for arbitrary $\mathcal{K}$.

Example Let $\mathcal{L} = -\Delta$, and denote by λ_j, $j = 0, 1, \cdots$, the corresponding eigenvalues, with $0 < \lambda_0 < \lambda_1 \leq \cdots$. Denote by v_j, $j = 0, 1, \cdots$, the associated eigenfunctions, which are assumed to be normalised and orthogonal. The principal eigenfunction v_0 is assumed to be positive. Consider the problem

$$-\Delta u + \varepsilon v_0 \int_{\Omega} v_1(y) u(y) dy = v_0 + \delta v_1, \tag{4}$$

where ε and δ are real constants (in the case where the second eigenvalue λ_1 is not simple, v_1 is assumed to be one of the associated eigenfunctions). This problem has one and only one solution, which is given by

$$u = \frac{1}{\lambda_0} \left(1 - \frac{\varepsilon \delta}{\lambda_1} \right) v_0 + \frac{\delta}{\lambda_1} v_1.$$

It is clear that it is possible to choose δ sufficiently small and ε sufficiently large, such that the function on the right–hand side of (4) is positive, while u changes sign – for ε sufficiently large, u will actually become negative. Furthermore, from the results in [10] it follows that the spectrum of the operator $\mathcal{L} - \varepsilon\mathcal{K}$ is the same as that of the operator $\mathcal{L}$ for all values of the parameter ε, and that the all v_j's are also eigenfunctions for the nonlocal problem, with the exception of v_1. In particular, the eigenfunction corresponding to the the first (positive) eigenvalue is still positive and equal to v_0. The second eigenvalue will now have an eigenfunction which is a linear combination of v_0 and v_1,

$$u_1 = \frac{\varepsilon}{\lambda_1 - \lambda_0} v_0 + v_1,$$

and which will also be positive for large enough ε.

One simple condition that ensures that a positivity result of the type mentioned is still applicable to (1), is when $\mathcal{K}$ is a positive operator, $\mathcal{K} \in L\left(C(\overline{\Omega}); L^q(\Omega)\right)$ with $q > \frac{1}{2}n$, and ε is small.

Definition 1 *Let X be a subspace of $L^1(\Omega)$. An operator $\mathcal{K} \in L\left(X; L^1(\Omega)\right)$ is called positive iff*

$$f \geq 0 \text{ implies } \mathcal{K}f \geq 0;$$

and strictly positive iff

$$0 \neq f \geq 0 \text{ implies } 0 \neq \mathcal{K}f \geq 0.$$

However, the result is still true for more general $\mathcal{K}$. We will show that it is possible for the class of (not necessarily positive) operators $\mathcal{K}$ that we consider to give a bound for ε such that $(\mathcal{L} - \varepsilon\mathcal{K})^{-1} \in L\left(C(\overline{\Omega}); C_0(\overline{\Omega})\right)$ is a positive operator. Needless to say that these results are uniform in f. Note that if $\mathcal{K}$ is bounded, then it easily follows that for every function f there exists ε_f for which the result holds.

This generalises the results in ([2]), where the kernel considered was taken to be separable. In particular, our results allow for K to be a Green's function associated with second order uniformly elliptic operators, and thus include the case of linear elliptic systems.

These results are based on pointwise estimates for the Green's function associated to the local operator $\mathcal{L}$.

2 The class of operators and the main result

We assume that $\mathcal{K}$ is the difference of two positive operators $\mathcal{K}_+, \mathcal{K}_- : L^1(\Omega) \to L^1(\Omega)$, that is $\mathcal{K} = \mathcal{K}_+ - \mathcal{K}_-$. If $\mathcal{K}$ is an integral operator we may define $\mathcal{K}_+, \mathcal{K}_-$ by the following kernels

$$\begin{aligned} K_+(x,y) &= \tfrac{1}{2}\left(K(x,y) + |K(x,y)|\right), \\ K_-(x,y) &= \tfrac{1}{2}\left(K(x,y) - |K(x,y)|\right). \end{aligned}$$

Assumption A $\mathcal{K}_+,\ \mathcal{K}_- \in L\left(C_0(\overline{\Omega}); L^q(\Omega)\right)$ *for some $q > n$.*

Let $\Psi : \overline{\Omega}^2 \times \overline{\Omega}^2 \to \mathbb{R}$ denote the function

$$\Psi(x,y;z,w) = \frac{|x-y|^{n-2}}{|x-z|^{n-2}\,|w-y|^{n-2}} \left(1 + \frac{|x-y|^2}{|x-z|\,|w-y|}\right).$$

We define a norm for K in a subclass of $L^1(\Omega^2)$ by

$$\|K\|_* := \sup_{x,y\in\Omega} \int_{z\in\Omega} \int_{w\in\Omega} |K(z,w)|\ \Psi(x,y;z,w)\, dw dz. \tag{5}$$

One might call an integral operator $\mathcal{K}$ with its kernel K satisfying $\|K\|_* < \infty$ to be of a double Kato-class type.

Assumption B *$\mathcal{K}_+$, $\mathcal{K}_-$ satisfy $\|\mathcal{K}_+\|_* < \infty$ and $\|\mathcal{K}_-\|_* < \infty$, where*

$$\|\mathcal{K}_\pm\|_* = \sup_{x,y\in\Omega} \int_{z\in\Omega} \mathcal{K}_\pm \left(\Psi\left(x,y;z,\cdot\right)\right)(z)\ dz. \tag{6}$$

Let us shortly discuss the influence of K_+ and of K_- by means of increasing ε in $(\mathcal{L}-\varepsilon\mathcal{K}_+)_0^{-1}$ and in $(\mathcal{L}-\varepsilon\mathcal{K}_-)_0^{-1}$, respectively. Here and in what follows, the index zero refers to the fact that we are considering homogeneous Dirichlet boundary conditions.

- *The positive part.*
  Since for $q > \frac{1}{2}n$ the operator $\mathcal{L}_0^{-1} \in L\left(L^q\left(\Omega\right);C_0(\overline{\Omega})\right)$ is compact we find that $\mathcal{L}_0^{-1}\mathcal{K}_+ \in L\left(C_0(\overline{\Omega});C_0(\overline{\Omega})\right)$ is a compact positive operator. Hence the spectral radius $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)$ is well defined. Let ϕ_0 denote the first eigenfunction of

  $$\left\{\begin{array}{rcll} \mathcal{L}\phi &=& \lambda\,\phi & \text{in } \Omega, \\ \phi &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{7}$$

  Since the strong maximum principle for the operator $\mathcal{L}$ implies that for any $u \in L^q\left(\Omega\right)$ with $q > n$, and hence for $u = \mathcal{K}_+\phi_0$, either $\mathcal{L}_0^{-1}\mathcal{K}_+\phi_0 = 0$, or $\mathcal{L}_0^{-1}\mathcal{K}_+\phi_0 \geq c\phi_0$, one finds that either $\mathcal{K}_+ \equiv 0$ on $C_0(\overline{\Omega})$ or $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right) > 0$. In the last case it follows by the Krein-Rutman Theorem that $\lambda_{0,\mathcal{K}_+} = \nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^{-1}$ is an eigenvalue for

  $$\left\{\begin{array}{rcll} \mathcal{L}\phi &=& \lambda\,\mathcal{K}_+\,\phi & \text{in } \Omega, \\ \phi &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{8}$$

  with $\phi_{0,\mathcal{K}_+} \in C_0(\overline{\Omega})$ a positive eigenfunction. The first eigenfunction $\phi_{0,\mathcal{K}_+}$ is assumed to be normalized by $\max\phi_{0,\mathcal{K}_+} = 1$. By the Krein-Rutman Theorem all other eigenvalues of (8) satisfy $\left|\lambda_{i,\mathcal{K}_+}\right| < \lambda_{0,\mathcal{K}_+}$.
  The effect of increasing ε (starting from 0) is that $(\mathcal{L}-\varepsilon\mathcal{K}_+)_0^{-1}$ becomes 'more' positive until it becomes singular for $\varepsilon = \lambda_{0,\mathcal{K}_+}$, that is, $\mathcal{L}-\varepsilon\mathcal{K}_+$ has a zero eigenvalue. The corresponding eigenfunction is the eigenfunction $\phi_{0,\mathcal{K}_+}$ of (8).

- *The negative part.*
  The effect of increasing ε (starting from 0) is that $(\mathcal{L}+\varepsilon\mathcal{K}_-)_0^{-1}$ becomes 'less' positive until positivity breaks down at some $\lambda_{c,\mathcal{K}_-}$. It means that for larger ε there are right hand sides that are positive but the corresponding solution is not. This number is in general not an eigenvalue. Note that for a local operator $\mathcal{K}_-$, that is $\left(\mathcal{K}_-u\right)(x) = a\left(x\right)u\left(x\right)$ with $a \in L^q\left(\Omega\right)$ and assuming $q > n$ and $a \geq 0$, one has $\lambda_{c,\mathcal{K}_-} = \infty$.

To state the main result we shall also need the notion of a strongly positive operator. Generally an operator $\mathcal{A} : C(\overline{\Omega}) \to C_0(\overline{\Omega})$ is called strongly positive if

$$f \in C(\overline{\Omega}) \text{ with } 0 \leq f \neq 0 \text{ implies } \mathcal{A}f\left(x\right) > 0 \text{ for all } x \in \Omega.$$

We will even show, assuming some more regularity, that the operators we are interested in behave near the boundary as in Hopf's boundary point lemma and will use strongly* positive in the following corresponding sense.

Definition 2 *We call an operator* $\mathcal{A} : C(\overline{\Omega}) \to C_0(\overline{\Omega})$ *strongly* positive if for every* $f \in C(\overline{\Omega})$ *with* $0 \leq f \neq 0$ *there exists* $c_f > 0$ *such that*

$$\mathcal{A}f(x) \geq c_f \phi_0(x) \text{ for all } x \in \overline{\Omega},$$

where ϕ_0 *is the first eigenfunction of* (7).
We will say $u \in C_0(\overline{\Omega})$ *satisfies*

$$u >^* 0$$

if there exists a constant $c > 0$ *such that* $u(x) \geq c\phi_0(x)$ *for all* $x \in \overline{\Omega}$.

In order to be able to use strongly* positive we assume $\mathcal{K}_\pm u \in L^q(\Omega)$ with $q > n$ for $u \in C_0(\overline{\Omega})$.

Theorem 3 *Let* Ω *and* $\mathcal{L}$ *be as above. Then there exists* $c_{\Omega,\mathcal{L}} > 0$ *such that the following holds. If* $\mathcal{K} = \mathcal{K}_+ - \mathcal{K}_-$, *with* $\mathcal{K}_+$ *and* $\mathcal{K}_-$ *positive operators satisfying Assumptions A and B , and*

$$\|\mathcal{K}_+\|_* + \|\mathcal{K}_-\|_* < c_{\Omega,\mathcal{L}}, \tag{9}$$

then $(\mathcal{L} - \mathcal{K})_0^{-1} : C(\overline{\Omega}) \to C_0(\overline{\Omega})$ *is strongly* positive.*

Remark: The formulation of this theorem uses a bound for the nonlocal term by a uniform estimate, that is, an estimate that is just depending on the $\|\cdot\|_*$-norm of $\mathcal{K}_+$ and $\mathcal{K}_-$. For fixed $\mathcal{K}_+$ and $\mathcal{K}_-$ in general much sharper estimates can be found. The more appropriate formulation is given as a corollary.

Corollary 4 *Let* Ω, $\mathcal{L}$, $\mathcal{K}_+$ *and* $\mathcal{K}_-$ *be as above. Then there exists* $c_{\Omega,\mathcal{L},\mathcal{K}_+} > 0$ *and* $c_{\Omega,\mathcal{L},\mathcal{K}_-} > 0$ *such that if*

$$\frac{t}{c_{\Omega,\mathcal{L},\mathcal{K}_+}} + \frac{s}{c_{\Omega,\mathcal{L},\mathcal{K}_-}} < 1,$$

then $(\mathcal{L} - t\mathcal{K}_+ + s\mathcal{K}_-)_0^{-1} : C(\overline{\Omega}) \to C_0(\overline{\Omega})$ *is strongly* positive.*

Remark: If $\mathcal{K}_+$ and $\mathcal{K}_-$ are local (multiplication) operators, then $c_{\Omega,\mathcal{L},\mathcal{K}_+}$ is the principal eigenvalue of (8) and $c_{\Omega,\mathcal{L},\mathcal{K}_-} = \infty$.

3 On the norm $\|\cdot\|_*$

First we will explain why this norm appears.

Lemma 5 *There exists a constant* λ_c *such that if* $\mathcal{K}$ *is a positive operator satisfying Assumptions A and B, then*

$$\mathcal{L}_0^{-1}\mathcal{K}\mathcal{L}_0^{-1} \leq \lambda_c \|\mathcal{K}\|_* \mathcal{L}_0^{-1}, \tag{10}$$

meaning that for all $f \in C(\overline{\Omega})$ *one finds*

$$f \geq 0 \text{ implies } \left(\mathcal{L}_0^{-1}\mathcal{K}\mathcal{L}_0^{-1}f\right) \leq \lambda_c \|K\|_* \left(\mathcal{L}_0^{-1}f\right).$$

Moreover, the spectral radius ν *of* $\mathcal{L}_0^{-1}\mathcal{K}$ *satisfies* $\nu \leq \lambda_c \|K\|_*$.

Proof. Let $G_{\Omega,\mathcal{L}}(x,y)$ denote the Green function for $\mathcal{L}_0^{-1}$, and suppose that $\mathcal{K}$ is an integral operator. The operator $Op = \mathcal{L}_0^{-1}\mathcal{K}\mathcal{L}_0^{-1}$ equals an integral operator with kernel

$$K_{Op}(x,y) = \int_{z\in\Omega}\int_{w\in\Omega} G_{\Omega,\mathcal{L}}(x,z)\, K(z,w)\, G_{\Omega,\mathcal{L}}(w,y)\, dzdw.$$

In order to prove (10) it is sufficient that for all $x,y \in \Omega$ one has

$$K_{Op}(x,y) \le \lambda_c \|K\|_* \, G_{\Omega,\mathcal{L}}(x,y).$$

We use the following two-sided estimate for a Green function on a bounded smooth domain in $\mathbb{R}^n$ with $n \ge 3$. See Zhao ([19]) for $\mathcal{L} = -\Delta$, Ancona ([1]) and Hueber-Sieveking ([13]) for the equivalence of 'arbitrary' Green functions (assuming some regularity conditions on $\mathcal{L}$), or [18]. There exist $c_1, c_2 > 0$ such that

$$c_1 \le \frac{G_{\Omega,\mathcal{L}}(x,y)}{|x-y|^{2-n}\min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)} \le c_2, \tag{11}$$

where d denotes the distance to the boundary $\partial\Omega$:

$$d(x) = \inf\{|x-y|\,; y \notin \Omega\}. \tag{12}$$

Now it is sufficient to prove that there exists a constant c such that for all $x,y \in \Omega$

$$\int_{z\in\Omega}\int_{w\in\Omega} \frac{\min\left(1, \frac{d(x)d(z)}{|x-z|^2}\right)\min\left(1, \frac{d(w)d(y)}{|w-y|^2}\right)}{|x-z|^{n-2}\,|w-y|^{n-2}} K(z,w)\, dzdw \le$$

$$\le c\,\|K\|_*\, |x-y|^{2-n} \min\left(1, \frac{d(x)\,d(y)}{|x-y|^2}\right). \tag{13}$$

The last estimate is equivalent with showing that

$$\sup_{x,y\in\Omega} \int_{z\in\Omega}\int_{w\in\Omega} \frac{|x-y|^{n-2}}{|x-z|^{n-2}|w-y|^{n-2}} \frac{\min\left(1, \frac{d(x)d(z)}{|x-z|^2}\right)\min\left(1, \frac{d(w)d(y)}{|w-y|^2}\right)}{\min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)} K(z,w)\ dzdw \le c\,\|K\|_*.$$

Using $d(z) \le d(x) + |x-z|$ and a similar estimate for $d(w)$ one finds

$$\frac{\min\left(1, \frac{d(x)d(z)}{|x-z|^2}\right)\min\left(1, \frac{d(w)d(y)}{|w-y|^2}\right)}{\min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)} \le 1 + \frac{|x-y|^2}{|x-z|\,|w-y|}$$

and hence (13) follows with $c = 1$. The estimate in the Lemma follows with $\lambda_c = c_2^2 c_1^{-1}$.

The last statement we prove by contradiction. Assume that $\nu > \lambda_c \|K\|_*$. Then the Krein-Rutman Theorem, since $\mathcal{L}_0^{-1}\mathcal{K}$ is positive, compact and has a positive spectral radius, implies that ν is the principal eigenvalue of $\mathcal{L}_0^{-1}\mathcal{K}$. The corresponding eigenfunction $\phi_{0,\mathcal{K}}$ is positive and hence

$$\nu\, \mathcal{L}_0^{-1}\mathcal{K}\phi_{0,\mathcal{K}} = \mathcal{L}_0^{-1}\mathcal{K}\mathcal{L}_0^{-1}\mathcal{K}\phi_{0,\mathcal{K}} \le \lambda_c \|K\|_* \mathcal{L}_0^{-1}\mathcal{K}\phi_{0,\mathcal{K}},$$

a contradiction. □

4 Proof of the main result

4.1 A formal asymptotic expansion

We set $\varepsilon = 1$ and we proceed formally first. Using the inverse operator of $\mathcal{L}$ with Dirichlet boundary condition we find that (1) is equivalent to

$$u = \mathcal{L}_0^{-1}\mathcal{K}u + \mathcal{L}_0^{-1}f \quad \text{or} \quad u = \left(\mathcal{I} - \mathcal{L}_0^{-1}\mathcal{K}\right)^{-1} \mathcal{L}_0^{-1}f.$$

If the series converges we have

$$u = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}\right)^k \mathcal{L}_0^{-1}f.$$

If $\mathcal{K}$ is a positive operator then a well defined series would imply that $(\mathcal{L} - \mathcal{K})_0^{-1}$ is a positive operator. If $\mathcal{K}_- \neq 0$ then the series might still be well defined but positivity is not obvious. Note that even the even powers $\left(\mathcal{L}_0^{-1}\mathcal{K}\right)^k$ do not have to be positive.

Remark: For $\mathcal{K}_+ = 0$ and $\mathcal{K}_- \neq 0$ the first two terms in the series $\sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}\right)^k$, namely $\mathcal{I} - \mathcal{L}_0^{-1}\mathcal{K}_-$ are not positive for any nonzero $\mathcal{K}_-$. Indeed, for $f \in C(\overline{\Omega})$ with $0 \neq f \geq 0$ and support $(f) \neq \Omega$ one finds by the strong maximum principle for $\mathcal{L}_0^{-1}$ that $\left(\mathcal{I} - \mathcal{L}_0^{-1}\mathcal{K}_-\right) f(x) < 0$ for $x \in \Omega \backslash$ support (f).

For a sign-changing or negative $\mathcal{K}$ we have to separate positive and negative part. Again assuming that the series converges it can be done as follows :

$$(\mathcal{L} - \mathcal{K})_0^{-1} = \left(\mathcal{I} + (\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right)^{-1} (\mathcal{L} - \mathcal{K}_+)_0^{-1} =$$

$$= \sum_{k=0}^{\infty} \left(-(\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right)^k (\mathcal{L} - \mathcal{K}_+)_0^{-1} =$$

$$= \left(\sum_{k=0}^{\infty} \left((\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right)^{2k}\right) \left(\mathcal{I} - (\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right) (\mathcal{L} - \mathcal{K}_+)_0^{-1}. \tag{14}$$

In order to show positivity for the solution operator $(\mathcal{L} - \mathcal{K})_0^{-1}$ it is hence sufficient to show that

- the series $\sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k$ converges;
- the series $\sum_{k=0}^{\infty} \left((\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right)^{2k}$ converges;
- the operator $\left(\mathcal{I} - (\mathcal{L} - \mathcal{K}_+)_0^{-1}\mathcal{K}_-\right) (\mathcal{L} - \mathcal{K}_+)_0^{-1}$ is positive.

4.2 The series for the positive part

Lemma 6 *Let λ_c be as in Lemma 5. If $\lambda_c \left\|\mathcal{K}_+\right\|_* < 1$, then*

1. $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right) < 1$ *and* $(\mathcal{L} - \mathcal{K}_+)_0^{-1} = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}$;
2. $(\mathcal{L} - \mathcal{K}_+)_0^{-1}$ *is strongly* positive and compact.*

Proof. We may suppose that $\mathcal{K}_+ \neq 0$. The series $\sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k$ converges if the spectral radius satisfies $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right) < 1$. Since the conditions imply that $\mathcal{L}_0^{-1}\mathcal{K}_+$ is positive and compact and has a positive spectral radius, the spectral radius equals the inverse of the first eigenvalue for (8). The assumption $\lambda_c \left\|\mathcal{K}_+\right\|_* < 1$ guarantees $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_+\right) < 1$. Hence

$$(\mathcal{L} - \mathcal{K}_+)_0^{-1} = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}$$

is well defined and $\mathcal{L}_0^{-1}$ being strongly* positive implies that $\sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}$ is strongly* positive. □

4.3 The series with the negative part

Lemma 7 *Let λ_c be as in Lemma 5. If*

$$\lambda_c \left\|\mathcal{K}_+\right\|_* + \lambda_c \left\|\mathcal{K}_-\right\|_* < 1,$$

then $\nu\left(\left(\mathcal{L} - \mathcal{K}_+\right)_0^{-1} \mathcal{K}_-\right) < 1$ and hence $\sum_{k=0}^{\infty} \left(\left(\mathcal{L} - \mathcal{K}_+\right)_0^{-1} \mathcal{K}_-\right)^{2k}$ is well defined and strictly positive.

Proof. We may assume that $\mathcal{K}_- \neq 0$ and have to distinguish two cases.

If $\mathcal{K}_+ = 0$ then the series converges if the spectral radius satisfies $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_-\right) < 1$. By Lemma 5 we find that $\nu\left(\mathcal{L}_0^{-1}\mathcal{K}_-\right) \leq \lambda_c \left\|K_-\right\|_*$ and we are done.

If $\mathcal{K}_+ \neq 0$ and $\lambda_c \left\|K_+\right\|_* < 1$ we use that

$$(\mathcal{L} - \mathcal{K}_+)_0^{-1} \mathcal{K}_- = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}\mathcal{K}_-.$$

Set $\nu^* = \nu\left(\left(\mathcal{L} - \mathcal{K}_+\right)_0^{-1} \mathcal{K}_-\right)$. Both $\mathcal{L}_0^{-1}\mathcal{K}_+$ and $\mathcal{L}_0^{-1}\mathcal{K}_-$ are positive operators. Moreover, $\mathcal{L}_0^{-1}\mathcal{K}_-$ is compact and since we have that for any positive function u either $\mathcal{L}_0^{-1}\mathcal{K}_- u >^* 0$ or $\mathcal{L}_0^{-1}\mathcal{K}_- u \equiv 0$ we find that there is a $c > 0$ such that $\mathcal{L}_0^{-1}\mathcal{K}_-\phi_0 > c\phi_0$ and hence that $\nu^* > 0$. By the Krein-Rutman Theorem ν^* is an eigenvalue with a, even the only, positive eigenfunction. Let ϕ^* denote the corresponding principal eigenfunction. Since $\phi^* > 0$ and since $\phi^* = \mathcal{L}_0^{-1}u$ with $u = \sum_{k=0}^{\infty} \left(\mathcal{K}_+\mathcal{L}_0^{-1}\right)^k \mathcal{K}_-\phi^* \in L^q(\Omega)$ and $u \geq 0$, we have that

$$\nu^*\phi^* = (\mathcal{L} - \mathcal{K}_+)_0^{-1} \mathcal{K}_-\mathcal{L}_0^{-1}u = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}\mathcal{K}_-\mathcal{L}_0^{-1}u \leq$$

$$\leq \sum_{k=0}^{\infty} \left(\lambda_c \left\|\mathcal{K}_+\right\|_*\right)^k \lambda_c \left\|\mathcal{K}_-\right\|_* \mathcal{L}_0^{-1}u = \frac{\lambda_c \left\|\mathcal{K}_-\right\|_*}{1 - \lambda_c \left\|\mathcal{K}_+\right\|_*}\phi^*.$$

Hence

$$\nu^* \leq \frac{\lambda_c \left\|\mathcal{K}_-\right\|_*}{1 - \lambda_c \left\|\mathcal{K}_+\right\|_*}.$$

The series in the lemma converges if

$$\frac{\lambda_c \left\|\mathcal{K}_-\right\|_*}{1 - \lambda_c \left\|\mathcal{K}_+\right\|_*} < 1$$

which is equivalent with

$$\lambda_c \left\|\mathcal{K}_-\right\|_* + \lambda_c \left\|\mathcal{K}_+\right\|_* < 1.$$

□

4.4 The operator is positive

Lemma 8 *Let λ_c be as in Lemma 5. If*

$$\lambda_c \left\|\mathcal{K}_+\right\|_* + \lambda_c \left\|\mathcal{K}_-\right\|_* < 1 \tag{15}$$

then $(\mathcal{L}-\mathcal{K})_0^{-1}$ is a strongly positive operator.*

Proof. Using the expression in (14) and the previous two lemmas it is sufficient to show that

$$\left(\mathcal{I} - (\mathcal{L}-\mathcal{K}_+)_0^{-1}\mathcal{K}_-\right)(\mathcal{L}-\mathcal{K}_+)_0^{-1}$$

is a stongly* positive operator. In other words, we have to show under the condition (15) that for all $u \in C(\overline{\Omega})$

$$u \geq 0 \text{ implies } (\mathcal{L}-\mathcal{K}_+)_0^{-1} u \geq (\mathcal{L}-\mathcal{K}_+)_0^{-1}\mathcal{K}_- (\mathcal{L}-\mathcal{K}_+)_0^{-1} u.$$

Strong* positivity will follow from the strict inequality in (15). Since

$$(\mathcal{L}-\mathcal{K}_+)_0^{-1} = \sum_{k=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}$$

we findwith Lemma 5 that

$$(\mathcal{L}-\mathcal{K}_+)_0^{-1}\mathcal{K}_- (\mathcal{L}-\mathcal{K}_+)_0^{-1} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \mathcal{L}_0^{-1}\mathcal{K}_- \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^m \mathcal{L}_0^{-1} \leq$$

$$\leq \lambda_c \left\|K_-\right\|_* \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \left(\mathcal{L}_0^{-1}\mathcal{K}_+\right)^k \left(\lambda_c \left\|K_+\right\|_*\right)^m \mathcal{L}_0^{-1} =$$

$$= \frac{\lambda_c \left\|K_-\right\|_*}{1-\lambda_c \left\|K_+\right\|_*} (\mathcal{L}-\mathcal{K}_+)_0^{-1}.$$

One concludes by using (15). □

5 Some examples

We start by considering local operators

$$\mathcal{K}u(x) = \varphi(x)\, u(x). \tag{16}$$

Lemma 9 *Let $p > n$. For $\mathcal{K}$ as in (16) we find that there is a constant c_Ω^p such that for all $\varphi \in L^p(\Omega)$:*

$$\left\|\mathcal{K}_\pm\right\|_* \leq c_\Omega^p \left\|\varphi_\pm\right\|_{L^p(\Omega)}.$$

Proof. Since for $q < \frac{n}{n-1}$ one has $\left\|\frac{1}{|\cdot - y|^{n-1}}\right\|_{L^q(\Omega)} \leq c_{\Omega,q}$ independently of y, we find straightforwardly by using

$$\frac{|x-y|}{|z-x||z-y|} \leq \frac{|x-z|+|z-y|}{|z-x||z-y|} = \frac{1}{|z-y|} + \frac{1}{|z-x|}$$

that for $\frac{1}{p}+\frac{1}{q}=1$

$$\|\mathcal{K}_\pm\|_* = \sup_{x,y\in\Omega} \int_{z\in\Omega} \varphi_\pm(z)\ \Psi(x,y;z,z)\ dz =$$

$$= \sup_{x,y\in\Omega} \int_{z\in\Omega} \varphi_\pm(z) \left(\tfrac{|x-y|^{n-2}}{|z-x|^{n-2}|z-y|^{n-2}} + \tfrac{|x-y|^{n}}{|z-x|^{n-1}|z-y|^{n-1}} \right) dz \leq$$

$$\leq c \sup_{x,y\in\Omega} \int_{z\in\Omega} \varphi_\pm(z) \left(\tfrac{1}{|z-y|^{n-2}} + \tfrac{1}{|z-x|^{n-2}} + \tfrac{|x-z|}{|z-y|^{n-1}} + \tfrac{|z-y|}{|z-x|^{n-1}} \right) dz \leq$$

$$\leq c'_{\Omega,q}\ \|\varphi_\pm\|_{L^p(\Omega)}$$

□

Let $\varphi \in L^p(\Omega)$ for some $p > n$. Using the lemma above one recovers Hopf's boundary point Lemma for

$$\begin{cases} -\Delta u - \varphi u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\|\varphi\|_{L^p(\Omega)}$ is small, but φ not necessarily bounded above pointwisely.

An important family of operators that are allowed are the following. By results of Grüter and Widman ([12]) one finds that a second order uniformly elliptic operator with Dirichlet boundary conditions and with quite general coefficients has a Green function satisfying

$$0 \leq G(x,y) \leq k\,|x-y|^{2-n}. \tag{17}$$

Also other homogeneous boundary conditions such as Neuman ($\frac{\partial}{\partial n}u = 0$) or Robin, assuming that the spectrum lies in the right half plane, have a Green function satisfying a similar estimate. Such an estimate can be proven similarly as in Theorem 4.6.11 of [5]. See also the remark on page 89 of [5].

Lemma 10 *If $\mathcal{K}$ is a integral operator with kernel such that for some $\varepsilon > 0$*

$$K(x,y) = |x-y|^{\varepsilon-n} M(x,y)$$

and $M \in L^p(\Omega^2)$ with $\dfrac{p}{n} > \max\left\{1, \dfrac{1}{\varepsilon}\right\}$, then for some constant $c_\Omega^{p,\varepsilon}$

$$\|K\|_* \leq c_\Omega^p\ \|M\|_{L^p(\Omega^2)}.$$

Proof. We will distinguish four areas:

I: $z,w\in\Omega$ such that $|z-x| \leq \frac{1}{3}|x-y|$ and $|w-y| \leq \frac{1}{3}|x-y|$;
II: $z,w\in\Omega$ such that $|z-x| \leq \frac{1}{3}|x-y|$ and $|w-y| > \frac{1}{3}|x-y|$;
III: $z,w\in\Omega$ such that $|z-x| > \frac{1}{3}|x-y|$ and $|w-y| \leq \frac{1}{3}|x-y|$;
IV: $z,w\in\Omega$ such that $|z-x| > \frac{1}{3}|x-y|$ and $|w-y| > \frac{1}{3}|x-y|$,

and in each of these four cases we will first estimate the explicite part of the integrand:

$$|z-w|^{\varepsilon-n} \left(\tfrac{|x-y|}{|x-z||w-y|} \right)^{n-2} \left(1 + \tfrac{|x-y|^2}{|x-z||w-y|} \right) = (*).$$

ad I. Using $|z-w| \geq \frac{1}{3}|x-y|$ and $|x-y|^2 \geq \frac{1}{9}|z-x|\,|w-y|$ we find:

$$(*) \leq 3^{n-\varepsilon}\left(\frac{|x-y|^{\varepsilon-2}}{|x-z|^{n-2}|w-y|^{n-2}}+\frac{|x-y|^{\varepsilon}}{|x-z|^{n-1}|w-y|^{n-1}}\right) \leq$$

$$\leq 3^{n-\varepsilon}\frac{10\,|x-y|^{\varepsilon}}{|x-z|^{n-1}|w-y|^{n-1}}.$$

ad II. Using $|w-y| \geq \frac{1}{3}|x-y|$ we find:

$$(*) \leq 3^{n-2}\frac{1}{|z-w|^{n-\varepsilon}|x-z|^{n-2}}+3^{n-1}\frac{|x-y|}{|z-w|^{n-\varepsilon}|x-z|^{n-1}}.$$

ad III. Using $|z-x| \geq \frac{1}{3}|x-y|$ we find:

$$(*) \leq 3^{n-2}\frac{1}{|z-w|^{n-\varepsilon}|w-y|^{n-2}}+3^{n-1}\frac{|x-y|}{|z-w|^{n-\varepsilon}|w-y|^{n-1}}.$$

ad IV. Using $|w-y| \geq \frac{1}{3}|x-y|$ and $|z-x| \geq \frac{1}{3}|x-y|$ we find:

$$(*) \leq 3^{n-2}\frac{10}{|z-w|^{n-\varepsilon}|x-z|^{n-2}}.$$

Since for q satisfying $q(\varepsilon-n) > -n$ and $q(1-n) > -n$, that is

$$q < \min\left\{\frac{n}{n-\varepsilon},\frac{n}{n-1}\right\},$$

we find that there exists $c_{q,\varepsilon,\Omega} > 0$ such that

$$\int_{z\in\Omega}\int_{w\in\Omega}\left(\frac{1}{|z-w|^{n-\varepsilon}|x-z|^{n-1}}\right)^q dwdz \leq$$

$$= \int_{z\in\Omega}|x-z|^{q(1-n)}\left(\int_{w\in\Omega}|z-w|^{q(\varepsilon-n)}\,dw\right)dz \leq c_{q,\varepsilon,\Omega}$$

independently of $x,y \in \Omega$. Similarly all other terms can be estimated and one finds by Hölder that

$$\|K\|_* = \sup_{x,y\in\Omega}\int_{z\in\Omega}\int_{w\in\Omega}|z-w|^{\varepsilon-n}|M(z,w)|\;\Psi(x,y;z,w)\,dwdz \leq c'_{q,\varepsilon,\Omega}\|M\|_{L^p(\Omega^2)}$$

for p such that $\frac{1}{p}+\frac{1}{q}=1$. The restriction on q implies that $p > \max\left\{1,\varepsilon^{-1}\right\}n$. □

Example 2 Consider the system

$$\begin{cases} -\Delta u + av &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\\ (-\Delta+1)\,v &= u \quad \text{in } \Omega,\\ \frac{\partial}{\partial n}v &= 0 \quad \text{on } \partial\Omega.\end{cases}$$

Let $p > n$. Then the following result holds.

If $\|a\|_{L^p(\Omega)}$ *is sufficiently small then* $0 \neq f \geq 0$ *implies* $u >^* 0$.

Note that there is no sign condition for a. This result follows by using $v = \mathcal{G}_N(u)$, where $\mathcal{G}_N(u)$ is the Green operator $(-\Delta+1)^{-1}_{\text{Neumann}}$. This operator has a kernel $G_N(x,y)$ satisfying

$$G_N(x,y) = g(x,y)\,|x-y|^{2-n}$$

with $g \in L^\infty(\Omega^2)$.

Example 3 Consider the biharmonic boundary value problem

$$\begin{cases} (\Delta)^2 u + g(\cdot, u, Du, \Delta u, D\Delta u) = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $Du = \left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$. Assume that $g \in C^1\left(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n\right)$ and that $g(\cdot,0,0,0,0) = 0$. Suppose that $u \in C^4\left(\bar{\Omega}\right)$ is a solution. Then the following holds.

If m is sufficiently small *and* $|g'|_\infty \leq m$ *then* $0 \neq f \geq 0$ *implies* $u >^* 0$.

It is shown as follows. Defining $v = -\Delta u$ we obtain

$$\begin{cases} -\Delta v + H(x) \cdot (\mathcal{G}v, D\mathcal{G}v, v, Dv) = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{G} = (-\Delta)_0^{-1}$, the Green function for the Poisson problem,

$$H(x) = \int_0^1 D_u g\left(x, su(x), sDu(x), s\Delta u(x), sD\Delta u(x)\right) ds$$

and D_u denotes differentiation with respect to the last 4 components. The bound on g' implies that H is small and hence that $f \geq 0$ implies $v \geq 0$. The classical maximum principle implies that $u = \mathcal{G}v \geq 0$.

In the next example we fix $\Omega = B_1(0) = \{x \in \mathbb{R}^n; |x| < 1\}$ and consider

$$\mathcal{K}u(x) = u(y)\, d\sigma_y \tag{18}$$

where $⨍$ is the average:

$$⨍_{|y|=|x|} u(y)\, d\sigma_y = \frac{\int_{|\theta|=1} u(|x|\,\theta)\, d\theta}{\int_{|\theta|=1} 1 d\theta}.$$

The operator $\mathcal{K}$ is artificial but it shows an operator that cannot be written by means of a kernel $K(x,y)$ on Ω^2. Before we state the result for this example we need the next lemma.

Lemma 11 *Let* $\mathcal{K}$ *be as in (18). Then* $\|\mathcal{K}\|_*$ *is bounded.*

Proof. Let us denote $\omega_n = \int_{|\theta|=1} 1 d\theta$. We have

$$\|\mathcal{K}\|_* = \sup_{x,y\in\Omega} \int_{z\in\Omega} \mathcal{K}\left(\Psi\left(x,y;z,\cdot\right)\right)(z)\ dz$$

$$= \omega_n^{-1} \sup_{x,y\in\Omega} \int_{z\in\Omega} \int_{|\theta|=1} \Psi\left(x,y;z,|z|\,\theta\right) d\theta dz =$$

$$= \omega_n^{-1} \sup_{x,y\in\Omega} \int_{z\in\Omega} \int_{|\theta|=1} \left(\frac{|x-y|^{n-2}}{|z-x|^{n-2}||z|\theta-y|^{n-2}} + \frac{|x-y|^{n}}{|z-x|^{n-1}||z|\theta-y|^{n-1}} \right) d\theta dz =$$

$$= \omega_n^{-1} \sup_{x,y\in\Omega} \int_{z\in\Omega} \left(\frac{|x-y|^{n-2}}{|z-x|^{n-2}|z|^{n-2}} \int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-2}} + \frac{|x-y|^{n}}{|z-x|^{n-1}|z|^{n-1}} \int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-1}} \right) dz. \tag{19}$$

We first estimate the two inner integrals:

$$\int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-2}} \le c \int_{s=0}^{1} \frac{s^{n-2}}{\left(s^2+\left|1-\frac{|y|}{|z|}\right|^2\right)^{\frac{n-2}{2}}} ds = (\mathrm{I}_i)$$

$$\int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-1}} \le c \int_{s=0}^{1} \frac{s^{n-2}}{\left(s^2+\left|1-\frac{|y|}{|z|}\right|^2\right)^{\frac{n-1}{2}}} ds = (\mathrm{II}_i)$$

Writing $R = \left|1-\frac{|y|}{|z|}\right|$ we obtain

$$(\mathrm{I}_i) = cR \int_{t=0}^{R^{-1}} \frac{t^{n-2}}{(t^2+1)^{\frac{n-2}{2}}} dt = \begin{cases} \mathcal{O}(1) & \text{for } R\in(0,1], \\ \mathcal{O}\left(R^{2-n}\right) & \text{for } R\in[1,\infty); \end{cases}$$

$$(\mathrm{II}_i) = c \int_{t=0}^{R^{-1}} \frac{t^{n-2}}{(t^2+1)^{\frac{n-1}{2}}} dt = \begin{cases} \mathcal{O}\left(\log R^{-1}\right) & \text{for } R\in\left(0,\frac{1}{2}\right], \\ \mathcal{O}\left(R^{1-n}\right) & \text{for } R\in\left[\frac{1}{2},\infty\right). \end{cases}$$

Hence the first part of (19) is estimated as follows

$$\omega_n^{-1} \sup_{x,y\in\Omega} \int_{z\in\Omega} \frac{|x-y|^{n-2}}{|z-x|^{n-2}|z|^{n-2}} \left(\int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-2}} \right) dz \le$$

$$\le c \sup_{x,y\in\Omega} \int_{z\in\Omega} \frac{|x-y|^{n-2}}{|z-x|^{n-2}|z|^{n-2}} \left(1+\frac{|y|}{|z|}\right)^{2-n} dz \le c \sup_{x,y\in\Omega} \int_{z\in\Omega} \frac{(|x-z|+|z|+|y|)^{n-2}}{|z-x|^{n-2}(|z|+|y|)^{n-2}} dz \le$$

$$\le c' \sup_{x,y\in\Omega} \int_{z\in\Omega} \left(\frac{1}{(|z|+|y|)^{n-2}} + \frac{1}{|z-x|^{n-2}} \right) dz < \infty.$$

The second part of (19) we split in two:

$$(\mathrm{II}_a) = \omega_n^{-1} \sup_{x,y\in\Omega} \int_{\substack{z\in\Omega \\ \left|1-\frac{|y|}{|z|}\right|\le\frac{1}{2}}} \frac{|x-y|^{n}}{|z-x|^{n-1}|z|^{n-1}} \left(\int_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-1}} \right) dz \le$$

$$\leq c \sup_{x,y\in\Omega} \int\limits_{\substack{z\in\Omega \\ \frac{2}{3}|y|\leq|z|\leq 2|y|}} \left(\tfrac{1}{|z|^{n-1}} + \tfrac{1}{|z-x|^{n-1}}\right) \log\left|1-\tfrac{|y|}{|z|}\right|^{-1} dz$$

and by Cauchy-Schwarz with $p = \frac{2n-1}{2n-2}$ and $q = 2n-1$ we find that the integral is bounded uniformly in x, y. Finally we find that the remaining integral in

$$(\mathrm{II}_b) = \omega_n^{-1} \sup_{x,y\in\Omega} \int\limits_{\substack{z\in\Omega \\ \left|1-\frac{|y|}{|z|}\right|\geq\frac{1}{2}}} \frac{|x-y|^n}{|z-x|^{n-1}|z|^{n-1}} \left(\int\limits_{|\theta|=1} \frac{d\theta}{\left|\theta-\frac{y}{|z|}\right|^{n-1}} \right) dz \leq$$

$$\leq c \sup_{x,y\in\Omega} \int\limits_{z\in\Omega} \frac{|x-y|^n}{|z-x|^{n-1}|z|^{n-1}} \left(1+\frac{|y|}{|z|}\right)^{1-n} dz \leq c \sup_{x,y\in\Omega} \int\limits_{z\in\Omega} \frac{(|x-z|+|z|+|y|)^{n-1}}{|z-x|^{n-1}(|z|+|y|)^{n-1}} dz$$

is bounded uniformly in x, y. □

Example 4 Consider

$$\begin{cases} -\Delta u + \varphi\mathcal{K}u &= f \quad \text{in } B_1(0), \\ u &= 0 \quad \text{on } \partial B_1(0), \end{cases}$$

with $\mathcal{K}$ as in (18). From the last lemma we find:

if $\|\varphi\|_{L^\infty(B_1(0))}$ *is sufficiently small then* $0 \neq f \geq 0$ *implies* $u >^* 0$.

Let $p > n$. The result even holds if $\|\varphi\|_{L^p(B_1(0))}$ is sufficiently small.

6 The associated eigenvalue problem

We now consider the special case where $\mathcal{K}$ is a bounded integral operator with a separable kernel of the form

$$(\mathcal{K}u)(x) = a(x) \int_\Omega b(y)u(y)dy,$$

and study the spectrum of the corresponding operator $\mathcal{L}u - \mathcal{K}u$ – here $\mathcal{L}$ will be assumed to be self–adjoint, so that the results from [10] can be applied directly. Note that $\mathcal{K}$ is not required to be self–adjoint, and so the resulting operator will not, in general, be self–adjoint either. Although this is a very particular case, it is quite important as it is the type of linear operator that will arise naturally when considering the linearized eigenvalue problem associated with a nonlocal reaction–diffusion equation of the type

$$u_t + \mathcal{L}u = f(u, \bar{u}),$$

where

$$\bar{u} = \int_\Omega g(x, u(x,t))dx.$$

The function $a(x)b(y)$ clearly falls into the class of kernels considered if, for instance, a and b are assumed to be continuous in $\bar{\Omega}$. We shall now consider the associated eigenvalue problem and give conditions for the existence of a positive eigenfunction associated with the smallest real eigenvalue. Existence of such an eigenfunction is ensured by the Krein–Rutman Theorem, provided that the operator has a compact inverse which is strongly positive. Thus Theorem 3 automatically implies that if the integral term has a sufficiently small norm then the first eigenvalue will have an associated positive eigenfunction. We shall now give a result which is independent of the norm of the nonlocal term, and which applies even when the operator is no longer positive.

Proposition 12 *Assume that the operator $\mathcal{L}$ is self–adjoint and consider the operator A defined by*

$$Au = \mathcal{L}u - a(x)\int_\Omega b(y)u(y)dy.$$

Let v_0 denote the first (positive) eigenfunction of (7). Then, if a is positive and

$$\int_\Omega b(y)v_0(y)dy > 0,$$

there will exist an odd number of real eigenvalues of A which are smaller than λ_0. Furthermore, the eigenfunctions associated with these eigenvalues can be chosen to be positive. In particular, there exists a first (not necessarilly simple) eigenvalue which is smaller than λ_0 and which will have associated with it a positive eigenfunction.

Proof. From the hypothesis we have that

$$a_0 b_0 = \int_\Omega a(x)v_0(x)dx \int_\Omega b(x)v_0(x)dx > 0.$$

It now follows from the results in [10] that the operator A will have an odd number of real eigenvalues $\lambda_{A,i} < \lambda_0$. Let λ be one of these eigenvalues and u_0 the corresponding eigenfunction. Then the operator $(\mathcal{L} - \lambda I)_0^{-1}$ will be positive, and as

$$\int_\Omega b(y)u_0(y)dy \neq 0$$

(for otherwise λ would have to be an eigenvalue of the local operator $\mathcal{L}$), it follows that u_0 can be chosen in order that the integral term is positive. This, together with the fact that $a(x)$ is positive, implies that $u_0 > 0$ in Ω. □

Remark: Clearly the argument is able to show that the eigenfunction can be chosen to be positive does not depend on the special form of the nonlocal term and can be applied once it is known that there exists in fact a real eigenvalue to the right of λ_0.

In the case where $\mathcal{K} = \mathcal{K}_+$, that is, the perturbing operator is positive, then the results in [16] imply the existence of a principal dominant eigenvalue λ_0, that is, a real simple eigenvalue whose eigenfunction is in the cone of positive functions, it is the only such eigenfunction, and all other eigenvalues will be to the right of λ_0. Here, because $\mathcal{K}$ is not necessarilly positive, there will not, in general, exist such an eigenvalue as the following example shows.

Example Consider the eigenvalue problem

$$\begin{cases} -u'' - \varepsilon \sum_{k=1}^{3} a_k \sin(k\pi x) \int_0^1 \sum_{k=1}^{3} b_k \sin(k\pi y)u(y)dy = \lambda u \\ u(0) = u(1) = 0 \end{cases} \tag{20}$$

Due to the form of the nonlocal term, all but the first three eigenvalues will remain constant for all real values of the parameter ε. To determine the dependence of these three eigenvalues on ε, we note that it is possible to reduce (20) to a finite–dimensional problem of the form

$$(D - \varepsilon\Gamma)v = \lambda v,$$

where

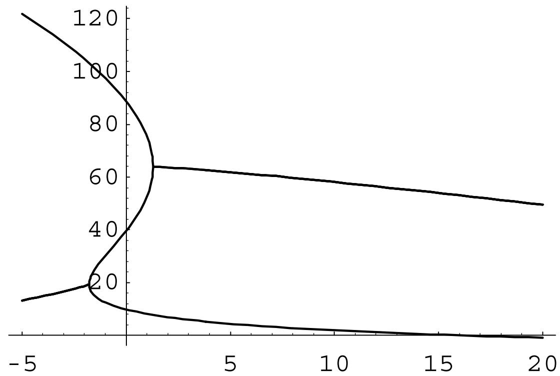
$$D = \begin{bmatrix} \pi^2 & 0 & 0 \\ 0 & 4\pi^2 & 0 \\ 0 & 0 & 9\pi^2 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

The pole–assignment Lemma (see [6], for the use of this result from control theory to this case), ensures that it is possible to place the eigenvalues of the matrix $D - \varepsilon\Gamma$ in the complex plane as desired (not counting multiplicities), by choosing the rank–1 matrix Γ appropriately. However, this result does not guarantee that it is possible to do so while keeping the function a in Proposition 12 positive. We now show that this is indeed possible and that as ε increases we go from one situation where there exists only one positive eigenfunction to the case where there are three eigenvalues to the left of the first eigenvalue of the local operator, and thus to a situation where there are three positive eigenfunctions.

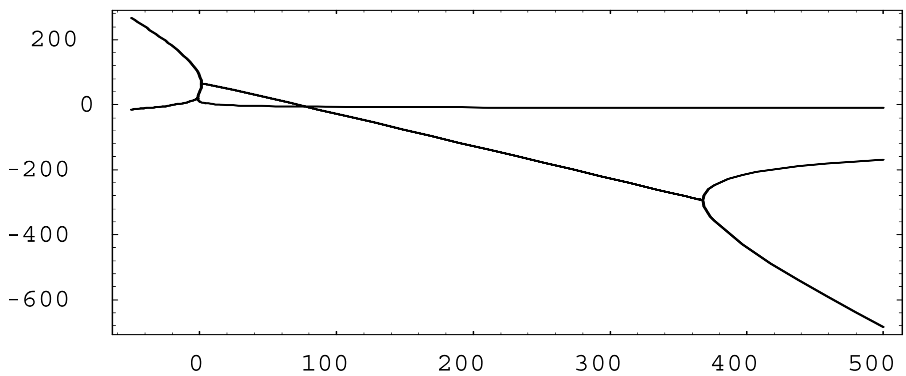
Let $(a_1, a_2, a_3) = (2, 1, 1)$ and $(b_1, b_2, b_3) = (1, -10, 10)$. Then

$$\Gamma = \begin{bmatrix} 2 & -20 & 20 \\ 1 & -10 & 10 \\ 1 & -10 & 10 \end{bmatrix}$$

and, as ε is increased from zero, the first three eigenvalues will start moving in the following way: $\lambda_0(\varepsilon)$ and $\lambda_2(\varepsilon)$ will decrease while $\lambda_1(\varepsilon)$ increases. If we continue increasing ε, at some point $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ will collide and become complex. If ε is increased further, then this pair of eigenvalues will become real again, but will now appear below $\lambda_0(\varepsilon)$ which is smaller than $\lambda_0(0)$. As the function $a(x)$ is positive for this choice of (a_1, a_2, a_3), it follows from the Remark after Proposition 12 that the three corresponding eigenfunctions will be positive.



The functions $\varepsilon \mapsto \mathrm{Re}\lambda_i(\varepsilon)$, $i = 1, 2, 3$.



The same but for a larger range of ε.

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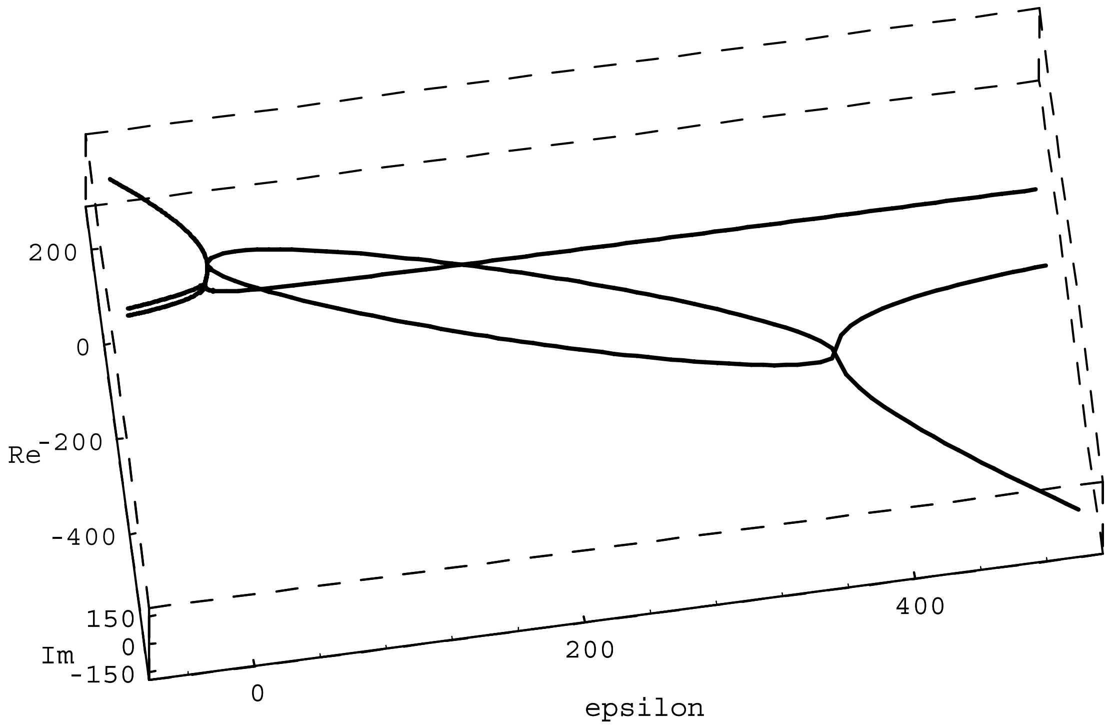

Figure 1: The functions $\lambda_i : \mathbb{R} \to \mathbb{C}$ of the last example.