Positivity for a Strongly Coupled Elliptic System by Green Function Estimates

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1. Examples

Consider the elliptic system

$$\left[\begin{array}{ll} -\Delta u = f - \varepsilon\, q.\nabla v & \text{in } \Omega, \\ -\Delta v = u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and with sufficient regularity for f, q, and Ω. We are interested in the question,

When does $f > 0$ imply $u > 0$?

Suppose Ω allows a Green function $G(\cdot, \cdot)$ for $-\Delta$. Then, with the notations

$$\mathcal{G}(f)(x) = \int_\Omega G(x, y) f(y)\, dy \quad \text{and} \quad \mathcal{D}(f)(x) = q(x).\nabla f(x),$$

the differential equations can be replaced by the following integral equation for u:

$$u(x) = \mathcal{G}(f)(x) - \varepsilon(\mathcal{G}\,\mathcal{D}\,\mathcal{G})(u)(x).$$

After solving for u we get v by $v(x) = \mathcal{G}(u)(x)$.

By exchanging the order of integration one finds

$$(\mathcal{G}\,\mathcal{D}\,\mathcal{G})(u)(x) = \int_\Omega \left(\int_\Omega G(x, z) q(z).\nabla_z G(z, y)\, dz \right) u(y)\, dy.$$

Then, at least formally, we can write

$$u = \left(I + \sum_{k=1}^{\infty} (-\varepsilon\, \mathcal{G}\,\mathcal{D}\,\mathcal{G})^k \right) \mathcal{G}\, f.$$

If one can show that there is $M \in \mathbb{R}$, such that, uniformly for $x \neq y$ in Ω, both

$$\int_{\Omega} G(x, z) G(z, y)\, dz \;\leq\; M\, G(x, y)$$

$$\left| \int_{\Omega} G(x, z) q(z).\nabla_z G(z, y)\, dz \right| \;\leq\; M\, G(x, y) \tag{2}$$

hold, then it follows for $\varepsilon < M^{-2}$ that the series converges pointwise. Moreover, from the positivity of the Green function we find for $\varepsilon < \frac{1}{2} M^{-2}$ that $u \geq 0$ if $f \geq 0$, and even $u > 0$ in Ω if $0 \neq f \geq 0$.

In a similar way, one may consider systems like

$$\begin{bmatrix} -\Delta u = f - \varepsilon\, q_1.\nabla v & \text{in } \Omega, \\ -\Delta v = q_2.\nabla u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{bmatrix}$$

or

$$\begin{bmatrix} -\Delta u = f - \varepsilon\, v & \text{in } \Omega, \\ -\Delta v = u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{bmatrix}$$

The simplest system where estimates like (2) are necessary for positivity is the following:

$$\begin{bmatrix} -\Delta u = f - \varepsilon(q.\nabla v + v) & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{bmatrix}$$

One finds $u = (\mathcal{G} - \varepsilon\, \mathcal{G}\,\mathcal{D}\,\mathcal{G} - \varepsilon\, \mathcal{G}^2) f$. For every $f > 0$ the function u is positive if $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ with $\varepsilon_0 = \frac{1}{2} M^{-1}$.

Remark. Notice that ε does not depend on f. It is straightforward from the strong maximum principle that there is positive ε_f (depending on f) such that u is positive for that $f > 0$. It is not clear whether one can use a compactness argument to obtain a uniform ε.

2. Introduction

We will consider elliptic systems of the type

$$
\left[
\begin{array}{ll}
L_1 u = f - \varepsilon\, g(., v, \nabla v) & \text{in } \Omega \\
L_2 v = f & \text{in } \Omega \\
u = v = 0 & \text{on } \partial\Omega
\end{array}
\right.
\tag{3}
$$

where L_1 and L_2 are two (possibly different) second-order elliptic operators. We suppose that for some functions $h(\cdot)$ and $k(\cdot)$ the following estimate holds:

$$
g(x, v, p) \le h(x)v + k(x)|p| \qquad \text{for all } (x, v, p) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^n.
\tag{4}
$$

The aim of this paper is to show the existence of a positive constant ε_0 such that u is positive whenever f is positive and $\varepsilon \in [0, \varepsilon_0)$.

For $n \ge 3$ we may use the functions h and k from appropriate Schechter-type spaces. For $n = 2$ we will use related spaces. See Simon in [17] for $\Omega = \mathbb{R}^n$.

For the proof we need pointwise estimates for the Green functions. The main estimate for $n > 2$ is an almost direct consequence of known results. These results were obtained by several authors with several regularity assumptions. For equivalence of Green functions and two-sided estimates, see the articles by Ancona [1], Hueber and Sieveking [14,15], Zhao [21], and Cranston, Fabes, and Zhao [8]. See also [19]. For the estimate on the gradient of the Green function, see the papers by Widman [20] and Cranston and Zhao [9].

For $n = 2$, estimates are obtained in [1,22]. The estimates are not sufficient to give the full result as for $n > 2$. We will derive the two-sided estimate we need for Green functions in two-dimensional domains. This we will only prove for elliptic operators with constant coefficients in front of the derivatives.

Second, by using the estimates for the Green functions, we will give elementary proofs of the so-called 3G-Theorem both for $n > 2$ and $n = 2$. For $n > 2$ see Cranston *e.a* in [8]. The 3G-Theorem gives bounds for $G(x, z)G(z, y)/G(x, y)$. By elementary means we will also derive a bound for $G(x, z)\nabla G(z, y)/G(x, y)$. Estimates of this quotient can also be found in [9].

In Section 3 we consider dimensions ≥ 3 and in Section 4 dimension 2. In Section 5 we show some relations with probability theory.

Some systems like (1), but where g does not depend on ∇v, so-called weakly coupled systems, have been studied in [18, 2, 3, 4 and 19]. One can use the result for (1) for more generally coupled systems. This is done in [11, 19, 4] for some classes of weakly coupled noncooperative systems.

Notations: $a \wedge b = \min(a, b), a \vee b = \max(a, b),$

the distance of x to $\partial\Omega$: $d_x = \inf\{|x - y|; y \in \partial\Omega\},$
the diameter of Ω: $D^\Omega = \sup\{|x - y|; x, y \in \Omega\}.$

By c_i we denote constants that are independent of x, y, or z. If necessary we suppose x, y, and z do not coincide.

3. In 3 and higher dimensions

3.1. Main result for $n \geq 3$.   We start with the regularity assumptions.

(a) L is a uniformly elliptic operator with Hölder-continuous coefficients, that is,

$$L = -\sum_{ij=1}^{n} a_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\cdot)\frac{\partial}{\partial x_i} + c(\cdot), \tag{5}$$

with for some $c_1, c_2 > 0$, $\gamma \in (0, 1]$:

$$c_1|\xi|^2 \leq \sum_{ij=1}^{n} a_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2 \text{ for all } x \in \overline{\Omega}, \ \xi \in \mathbb{R}^n,$$

$$a_{ij}(\cdot), b_i(\cdot), c(\cdot) \in C^{0,\gamma}(\overline{\Omega}).$$

Moreover, assume that $c(\cdot) \geq 0$.

(b)   The domain Ω is an open, bounded, and connected subset of $\mathbb{R}^n$ and $\partial\Omega \in C^{1,1}$.

(c)   Define for $\vartheta \in (0, n]$ the norm

$$\|h\|_{\vartheta,1} = \sup_{x\in\Omega} \int_\Omega |x - y|^{\vartheta-n}|h(y)|\,dy, \tag{6}$$

and $h \in \mathcal{M}_{\vartheta,1}$ if and only if $\|h\|_{\vartheta,1} \leq \infty$. The space $\mathcal{M}_{\vartheta,1}$ is related with the Schechter spaces, which are defined on $\mathbb{R}^n$. See Definition A.15 of [17].

If $p \in (n\,\vartheta^{-1}, \infty]$, then $L_p(\Omega) \subset \mathcal{M}_{\vartheta,1}$.

Theorem 3.1.   *Suppose L_1, L_2, Ω have the above regularity. Suppose that* (4) *holds with* $h \in \mathcal{M}_{2,1}$ *and* $k \in \mathcal{M}_{1,1}$.

Then there is $\varepsilon_0 > 0$, such that for all $\varepsilon \in [0, \varepsilon_0)$ and for all nonzero $f \geq 0$ one finds that the solution u of (3) *satisfies $u(x) > 0$ in Ω.*

Remark. The proof yields $u_{(\varepsilon)} \geq (1 - \varepsilon/\varepsilon_0)u_{(\varepsilon)}$, and hence a strong minimum principle, that is, $-(\partial/\partial n)u > 0$ at $\partial\Omega$, with n the outward normal.

3.2. Known estimates for $n \geq 3$.

By Hueber and Sieveking in [14,15] and Zhao in [21] there are $c_1, c_2 > 0$ such that the Green function G_L, for an elliptic operator L as above, satisfies

$$c_1|x - y|^{2-n}\left(1 \wedge \frac{d_x d_y}{|x - y|^2}\right) \leq G_L(x, y) \leq c_2|x - y|^{2-n}\left(1 \wedge \frac{d_x d_y}{|x - y|^2}\right) \tag{7}$$

for all $x, y \in \Omega$.

By a theorem of Widman in [20] there are $c_1, c_2 > 0$ such that

$$G_L(x, y) \leq c_1|x - y|^{2-n}\left(1 \wedge \frac{d_x}{|x - y|} \wedge \frac{d_y}{|x - y|}\right) \tag{8}$$

and

$$|\nabla_x G_L(x, y)| \leq c_2|x - y|^{1-n}\left(1 \wedge \frac{d_y}{|x - y|}\right) \tag{9}$$

for all $x, y \in \Omega$. Estimate (8) can also be obtained from the right-hand side of (7).

3.3. 3G type theorems for $n \geq 3$.

In [8] Cranston, Fabes, and Zhao show a bound for $G(x, z)G(z, y)/G(x, y)$ in what they call the 3G Theorem. Using the result of [14,15] we need more regularity of the boundary, but we will also obtain a stronger estimate that might be interesting in itself.

Lemma 3.2. *Suppose $G_i(\cdot, \cdot), i = 1, 2, or 3$, satisfies (7) and (8). Then there is a constant M_1 such that for $0 \leq \tau \leq n - 2$ and disjoint $x, y, z \in \Omega$:*

$$\frac{G_1(x, z)G_2(z, y)}{G_3(x, y)} \leq M_1\left(|x - z|^{2-n-\tau} + |y - z|^{2-n-\tau}\right)|x - y|^\tau. \tag{10}$$

Corollary 3.3. *With G_i as above, one finds for $h \in \mathcal{M}_{\vartheta,1}$, with $\vartheta \in (0, 2]$ if $n \geq 4$, and $\vartheta \in [1, 2]$ if $n = 3$, that*

$$\left|\int_\Omega \frac{G_1(x, z)h(z)G_2(z, y)}{G_3(x, y)}\,dz\right| \leq c_\vartheta M_1\|h\|_{\vartheta,1}|x - y|^{2-\vartheta} \quad \text{for all } x \neq y \in \Omega. \tag{11}$$

Proof. First assume that $z \in \Omega_1 := \{z \in \Omega; |x - z| \leq |y - z|\}$. For $z \in \Omega_1$ we find $|x - y| \leq |x - z| + |z - y| \leq 2|y - z|$.

If $|x - y| > \frac{1}{2}(d_x \vee d_y)$ we get from (7) and (8) that for all $z \in \Omega$

$$\frac{G_1(x, z)G_2(z, y)}{G_3(x, y)} \leq c|x - z|^{2-n}\left(1 \wedge \frac{d_x}{|x - z|}\right)\frac{|z - y|^{-n}}{|x - y|^{-n}}\frac{d_z\, d_y}{d_x\, d_y}$$

$$\leq c|x - z|^{2-n}\frac{2d_z}{d_x + |x - z|}\frac{|x - y|^n}{|z - y|^n}$$

$$\leq 2c|x - z|^{2-n}\frac{|x - y|^n}{|z - y|^n}. \qquad (12)$$

Now assume that $|x - y| \leq \frac{1}{2}(d_x \vee d_y)$ and from $|d_x - d_y| \leq |x - y|$ it follows that $|x - y| \leq d_x \wedge d_y$. Using (7) and (8) we get the estimate

$$\frac{G_1(x, z)G_2(z, y)}{G_3(x, y)} \leq c|x - z|^{2-n}\frac{|x - y|^{n-2}}{|z - y|^{n-2}} \quad \text{for all } x, y, z \in \Omega. \qquad (13)$$

We find (10) for $0 \leq \tau \leq n - 2$ and $z \in \Omega_1$, since

$$\frac{|x - y|^{n-2}}{|z - y|^{n-2}} \leq \frac{|x - y|^{\tau}}{|z - y|^{\tau}}\frac{|x - y|^{n-2-\tau}}{|z - y|^{n-2-\tau}} \leq \frac{|x - y|^{\tau}}{|z - x|^{\tau}}2^{n-2-\tau}. \qquad (14)$$

Similar estimates hold with x replaced by y, and we get (10) for $z \in \Omega\backslash\Omega_1$. $\square$

Lemma 3.4. *Suppose $G_1(\cdot, \cdot)$ and $G_3(\cdot, \cdot)$ satisfy (7) and (8), and $G_2(\cdot, \cdot)$ satisfies (9). Then there is a constant M_2 such that for $0 \leq \tau \leq n - 2$ and $x, y, z \in \Omega$:*

$$\frac{G_1(x, z)|\nabla_z G_2(z, y)|}{G_3(x, y)} \leq M_2\left(|x - z|^{1-n-\tau} + |y - z|^{1-n-\tau}\right)|x - y|^{\tau}. \qquad (15)$$

Estimates with $\tau = 0$ in Lemma 3.2 and 3.4 above can be found in [9].

Corollary 3.5. *With G_i as above one finds for $k \in \mathcal{M}_{\vartheta,1}$, with $\vartheta \in (0, 1]$ if $n \geq 4$, and $\vartheta = 1$ if $n = 3$, that*

$$\int_{\Omega}\frac{G_1(x, z)|k(z)||\nabla G_2(z, y)|}{G_3(x, y)}\, dz \leq c_{\vartheta}M_2\|k\|_{\vartheta,1}|x - y|^{1-\vartheta} \quad \text{for all } x \neq y \in \Omega. \qquad (16)$$

Proof. Using (8) and (9) one finds for $|x - y| > \frac{1}{2}(d_x \vee d_y)$ that

$$\frac{G_1(x, z)|\nabla_z G_2(z, y)|}{G_3(x, y)}$$

$$\leq c\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{-n}d_x \, d_y}\left(\frac{d_x}{|x - z|} \wedge \frac{d_x d_z}{|x - z|^2}\right)\left(1 \wedge \frac{d_y}{|z - y|}\right) \qquad (17)$$

for all $z \in \Omega$, and if $|x - y| \leq \frac{1}{2}(d_x \vee d_y)$ that

$$\frac{G_1(x, z)|\nabla_z G_2(z, y)|}{G_3(x, y)} \leq c\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{2-n}} \qquad \text{for all } z \in \Omega. \qquad (18)$$

Define Ω_1 as previously. For $z \in \Omega_1$ (17) can be estimated for $0 \leq \tau \leq n$ by

$$\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{-n}} \frac{1}{|x - z|} \frac{1}{|z - y|} \leq \frac{|x - y|^n}{|x - z|^{n-1}|z - y|^n} \leq \frac{2^n}{|x - z|^{n-1}} \frac{|x - y|^\tau}{|x - z|^\tau} \qquad (19)$$

For $z \in \Omega_1$ and $0 \leq \tau \leq n - 2$ we get for (18)

$$\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{2-n}} \leq \frac{|x - y|^{n-2}}{|x - z|^{n-2}|z - y|^{n-1}} \frac{|z - y|}{|x - z|} \leq \frac{2^{n-2}}{|x - z|^{n-1}} \frac{|x - y|^\tau}{|x - z|^\tau} \qquad (20)$$

For $z \in \Omega \backslash \Omega_1, 0 \leq \tau \leq n$, and using $d_z \leq d_y + |z - y|$, (17) is estimated by

$$\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{-n}} \frac{d_z}{|x - z|^2}\left(\frac{1}{d_y} \wedge \frac{1}{|z - y|}\right) \leq \frac{|x - y|^n}{|x - z|^n|z - y|^{n-1}} \frac{2d_z}{d_y + |z - y|}$$

$$\leq \frac{2^{n+1}}{|y - z|^{n-1}} \frac{|x - y|^\tau}{|y - z|^\tau}. \qquad (21)$$

For $z \in \Omega \backslash \Omega_1, 0 \leq \tau \leq n - 2$, (18) is estimated by:

$$\frac{|x - z|^{2-n}|z - y|^{1-n}}{|x - y|^{2-n}} \leq \frac{2^{n-2}}{|y - z|^{n-1}} \frac{|x - y|^\tau}{|y - z|^\tau} \qquad \square \qquad (22)$$

3.4. The proof of Theorem 3.1. We can write the solution of (3) as

$$u(x) = \int_\Omega G_1(x, y)f(y) \, dy - \varepsilon \int_\Omega G_1(x, z)g(z, v(z), \nabla v(z)) \, dz. \qquad (23)$$

By condition (4) and using the positivity of f, $G(\cdot, \cdot)$ and hence of v we find that

$$
\begin{aligned}
u(x) \;\geq\; & \int_\Omega G_1(x, y) f(y)\, dy - \varepsilon \int_\Omega G_1(x, z)(h(z)v(z) + k(z)|\nabla v(z)|)\, dz \\[2mm]
\geq\; & \int_\Omega G_1(x, y) f(y)\, dy \\[1mm]
& - \varepsilon \int_\Omega G_1(x, z) \int_\Omega (h(z) G_2(z, y) + k(z)|\nabla_z G_2(z, y)|) f(y)\, dy\, dz \\[2mm]
\geq\; & \int_\Omega G_1(x, y) \left(1 - \varepsilon \int_\Omega \frac{G_1(x, z)}{G_1(x, y)} (h(z) G_2(z, y) + k(z)|\nabla_z G_2(z, y)|)dz\right) f(y)\, dy \\[2mm]
\geq\; & \int_\Omega G_1(x, y)(1 - \varepsilon M) f(y)\, dy > 0 \text{ for } \varepsilon \in [0, \varepsilon_0), x \in \Omega, \qquad (24)
\end{aligned}
$$

and with $M = c_2 M_1 \|h\|_{2,1} + c_1 M_2 \|k\|_{1,1}$, $\varepsilon_0 = M^{-1}$.

The last step follows from the assumptions on h and k and Corollaries 3.3 and and 3.4. $\qquad\square$

4. In 2 dimensions

4.1. Main result for $n = 2$. Again start with the regularity assumptions.

(a) L is an elliptic operator with

$$
L = -\sum_{ij=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} + c(\cdot), \qquad (25)
$$

$$
\frac{1}{4} |(a_{ij})^{-1} b|^2 \leq c(\cdot) \in C(\overline{\Omega}).
$$

(b)   The domain Ω is an open, bounded, and connected subset of $\mathbb{R}^2$ and $\partial\Omega \in C^{1,\gamma}$.

(c)   Define for $\vartheta \in (0, 2]$ the norm

$$
\|h\|_{\vartheta,1}^* = \sup_{x \in \Omega} \int_\Omega \log\left(\frac{eD^\Omega}{|x - y|}\right) |x - y|^{\vartheta-2} |h(y)|\, dy, \qquad (26)
$$

and $h \in \mathcal{M}_{\vartheta,1}^*$ if and only if $\|h\|_{\vartheta,1}^* \leq \infty$. Compare with Definition A.15' of [17]. If $p \in (2\vartheta^{-1}, \infty]$, then $L_p(\Omega) \subset \mathcal{M}_{\vartheta,1}^*$.

Theorem 4.1.   *Suppose L_1, L_2, Ω have the above regularity. Suppose that (4) holds with $h \in \mathcal{M}_{2,1}^*$ and $k \in \mathcal{M}_{1,1}^*$.*

Then there is $\varepsilon_0 > 0$, such that for all $\varepsilon \in [0, \varepsilon_0)$ and for all nonzero $f \geq 0$ one finds that the solution u of (3) *satisfies* $u(x) > 0$ *in* Ω.

Remark. In case that the function g in (3) satisfies

$$g(x, v, p) \leq c\, v \text{ for all } (x, v, p) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^2$$

we may combine the result of Ancona in [1] and the estimate of Zhao in [22] and find the result of Theorem 4.1 for uniform elliptic operators with nonconstant coefficients as in dimensions larger than 2.

4.2. Estimates of the Green function for $-\Delta$ when $n = 2$.

From the explicit Green function for $-\Delta$ on the unit ball B in $\mathbb{R}^2$, namely

$$G_B(x, y) = (4\pi)^{-1} \log\left(1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}\right), \tag{27}$$

we find the following estimate (with $c_1 = 1$ and $c_2 = 4$ if $\Omega = B$):

$$(4\pi)^{-1} \log\left(1 + c_1 \frac{d_x d_y}{|x - y|^2}\right) \leq G_\Omega(x, y) \leq (4\pi)^{-1} \log\left(1 + c_2 \frac{d_x d_y}{|x - y|^2}\right). \tag{28}$$

Lemma 4.2. *With Ω a domain in $\mathbb{R}^2$ as in (b), the Green function on Ω for $-\Delta$ satisfies* (28).

Using this result for the Laplacian we will prove the estimate for L, with the assumptions above, in Section 4.5.

Proof. (i) First assume that Ω is a simply connected domain. We will apply an idea that Riemann used for the mapping theorem named after him. See [12, p. 399] or [5]. Let $x_0 \in \Omega$ and set $u(x) = 2\pi\, G_\Omega(x, x_0)$. Then $u(\cdot)$ is harmonic in $\Omega \backslash \{x_0\}$ and the extension of $u(\cdot) + \log | \cdot - x_0|$ is harmonic in Ω.

Moreover, $\nabla u \neq 0$ in $\Omega \backslash \{x_0\}$ since Ω is simply connected. Indeed, if $\nabla u(x^*) = 0$, then either the set $\Omega_1 = \{x \in \Omega; u(x) > u(x^*)\}$ or the set $\Omega_2 = \{x \in \Omega; u(x) < u(x^*)\}$ has at least two components. The maximum principle shows that a component of Ω_1 contains x_0, and hence that there is only one component. So Ω_2 has at least two components. Since the boundary of every component of Ω_2 contains part of $\partial\Omega$ and $\partial\Omega$ is contained in $\partial\Omega_2$, $\partial\Omega$ has at least two components. Hence Ω cannot be simply connected: a contradiction.

Since $\partial\Omega$ is $C^{1,\gamma}$ one finds $u \in C^{1,\gamma}(\overline{\Omega} \backslash \{x_0\})$ [12, Th. 8.34]. The strong maximum principle implies that $\frac{\partial u}{\partial n} \neq 0$ on $\partial\Omega$, and hence $\nabla u \neq 0$ in $\overline{\Omega} \backslash \{x_0\}$. Fix $y_0 \in \Omega \backslash \{x_0\}$ and define v the

harmonic conjugate of u (defined by the Cauchy-Riemann equations) with $v(y_0) = 0$; that is,

$$v(x) = \int_\Gamma (-u_t(s,t)\,ds + u_s(s,t)\,dt),$$

where Γ is a curve in $\Omega \backslash \{x_0\}$ from y_0 to x; v is defined up to a multiple of 2π. After identifying $\mathbb{R}^2$ with C one finds that $f(z) = e^{-u(z)-iv(z)}$ is holomorphic in Ω and maps Ω conformally on the unit ball B. From the properties of u it follows that $f'(z)$ is well defined, continuous, and nonzero on $\overline{\Omega}$. Moreover, there is $c > 0$ such that

$$c\,d(z, \partial\Omega) \leq d(f(z), \partial B) \leq c^{-1} d(z, \partial\Omega) \quad \text{for } z \in \Omega,$$
$$c|z_1 - z_2| \leq |f(z_1) - f(z_2)| \leq c^{-1}|z_1 - z_2| \quad \text{for } z_1, z_2 \in \Omega.$$

Since $G_\Omega(x, y) = G_B(f(x), f(y))$, one finds that both sides of (28) hold for $G_\Omega(\cdot, \cdot)$.

(ii) The estimate from above for doubly connected domains.

We start with the annulus in $\mathbb{R}^2$, $A = \{1 < |x| < 2\}$.

Set $A_1 = \{x \in A; |x| < \frac{4}{3}\}$ and $A_2 = \{x \in A; \frac{5}{3} < |x|\}$.

Lindelöf's principle shows that

$$G_A(x, y) \leq G_{B^c}(x, y) = G_B(x|x|^{-2}, y|y|^{-2}) = G_B(x, y) \text{ for } x, y \in A\backslash A_2 \qquad (29)$$

$$G_A(x, y) \leq G_{2B}(x, y) = G_B\left(\frac{1}{2}x, \frac{1}{2}y\right) \quad \text{for } x, y \in A\backslash A_1. \qquad (30)$$

Since $d(z, \partial B) \leq 2d(z, \partial A)$ for $z \in A\backslash A_2$ and $d(z, \partial 2B) \leq 2d(z, \partial A)$ for $z \in A\backslash A_1$, the estimate follows.

For $x \in A_1$ and $y \in A_2$ (and vice versa) we use that $G_A(x, y)$ is a bounded harmonic function on $A_1 \times A_2$. By the maximum principle

$$G_A(x, y) \leq c\,\phi_0(x, y) \quad \text{for } (x, y) \in A_1 \times A_2, \qquad (31)$$

where $\phi_0(\cdot, \cdot)$ is the first eigenfunction on $A \times A$. Notice that $\phi_0(x, y) = \varphi_0(x)\varphi_0(y)$ with $\varphi_0(\cdot)$ the first eigenfunction on A. The estimate follows since $\varphi_0(x) \leq c\,d(x, \partial A)$ and $|x - y| > 1/3$.

For general doubly connected domains Ω one defines a conformal mapping f from Ω to A that is C^1 on $\overline{\Omega}$ and $f' \neq 0$ on $\overline{\Omega}$. Indeed, if Γ_1 and Γ_2 denote the two disjoint parts of the boundary, let u be the harmonic function on Ω that satisfies $u = 0$ on Γ_1 and $u = \log(2)$ on Γ_2, and define f as above. The upper estimate follows as in (i).

(iii) Let Ω be multiply connected. Since Ω is bounded and has a smooth boundary, there are at most finitely many holes. Let $S_1, \ldots, S_k$ denote the (closed) bounded components of Ω^c. Define the positive constant $\delta = \min\{d(S_i, S_j); 1 \leq i < j \leq k\}$ and set

$$\Omega_i = \left\{x \in \Omega; d(x, S_i) < \frac{1}{3}\delta\right\}.$$

We have to distinguish two cases: (1) $x \in \Omega_i$ and $y \in \Omega_j$ with $i \neq j$ and (2) otherwise.

In the first case $|x - y| > \frac{1}{3}\delta$ and the result follows from a similar argument as the one that used (31). Replace $A_1 \times A_2$ by $\Omega_i \times \Omega_j$.

In the second case there is an index i such that $d(x, S_j) \geq \frac{1}{3}\delta$ and $d(y, S_j) \geq \frac{1}{3}\delta$ for all $j \neq i$. Define the doubly connected domain $\Omega^* = \Omega \cup \{S_j; j \neq i\}$. Notice that there is $c > 0$ such that $d(z, \partial\Omega^*) \leq c\, d(z, \partial\Omega)$ for $z \in \Omega$ with $d(z, S_j) > \frac{1}{3}\delta$ for all $j \neq i$. Then by Lindelöf's principle the estimate follows from $G_\Omega(x, y) \leq G_{\Omega^*}(x, y)$.

(iv) The lower estimate for multiply connected domains can be obtained as follows.

Notice that there are simply connected domains Ω_1, Ω_2, and Ω_3 with $C^{1,\gamma}$ boundary
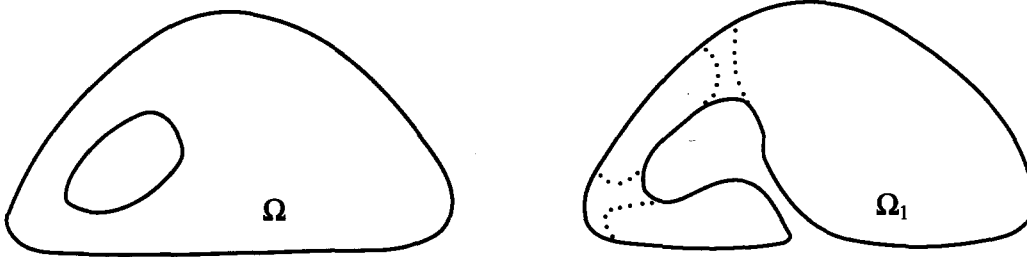
and such that $\Omega = \Omega_1 \cup \Omega_2 = \Omega_1 \cup \Omega_3 = \Omega_2 \cup \Omega_3$. Moreover, we can take these subdomains such that $\Omega \backslash \Omega_1$, $\Omega \backslash \Omega_2$, and $\Omega \backslash \Omega_3$ are seperated with a positive distance, say δ. Then for every pair $x, y \in \Omega$ there is $i \in \{1, 2, 3\}$ such that $x, y \in \Omega_i, d(x, \Omega \backslash \Omega_j) > \frac{1}{2}\delta$, and $d(y, \Omega \backslash \Omega_j) > \frac{1}{2}\delta$ for $j \neq i$. The estimate follows from $G_\Omega(x, y) \geq G_{\Omega_i}(x, y)$. $\square$

We will end this section with a technical lemma.

Lemma 4.3.

$$\frac{1}{4}\left(\log\left(1+\frac{d_x}{|x-y|}\right)+\log\left(1+\frac{d_y}{|x-y|}\right)\right)\wedge\frac{1}{4}\frac{d_xd_y}{|x-y|^2}$$

$$\leq\log\left(1+\frac{d_xd_y}{|x-y|^2}\right)$$

$$\leq 2\left(\log\left(1+\frac{d_x}{|x-y|}\right)\wedge\log\left(1+\frac{d_y}{|x-y|}\right)\wedge\frac{d_xd_y}{|x-y|^2}\right).\qquad(32)$$

Notice that the last expression can be estimated from above by a constant times the first expression.

Proof. Let $x^*\in\partial\Omega$ such that $|x-x^*|=d_x$. From $d_y\leq|y-x^*|\leq d_x+|y-x|$ it follows that

$$1+\frac{d_xd_y}{|x-y|^2}\leq 1+\frac{d_x(d_x+|y-x|)}{|x-y|^2}\leq\left(1+\frac{d_x}{|x-y|}\right)^2.$$

Hence

$$\log\left(1+\frac{d_xd_y}{|x-y|^2}\right)\leq 2\,\log\left(1+\frac{d_x}{|x-y|}\right).$$

One finds the second inequality of (32) by replacing x with y and from the inequality $\log(1+a)\leq a$ for $a\geq 0$.

To prove the first inequality of (32) we have to distinguish two cases.

(i) $|x-y|\leq\frac{1}{2}(d_x\vee d_y)$ and hence $|x-y|\leq d_x\wedge d_y$. It follows that

$$\log\left(1+\frac{d_xd_y}{|x-y|^2}\right)\geq\log\left(1+\frac{1}{4}\left(1+\frac{d_x}{|x-y|}\right)\left(1+\frac{d_y}{|x-y|}\right)\right)$$

$$\geq\frac{1}{4}\log\left(1+\left(1+\frac{d_x}{|x-y|}\right)\left(1+\frac{d_y}{|x-y|}\right)\right)$$

$$\geq\frac{1}{4}\left(\log\left(1+\frac{d_x}{|x-y|}\right)+\log\left(1+\frac{d_y}{|x-y|}\right)\right).$$

(ii) $|x-y|\geq\frac{1}{2}(d_x\vee d_y)$. Since $\log(1+a)\geq\frac{1}{4}a$ for $a\in[0,4]$ one finds

$$\log\left(1+\frac{d_xd_y}{|x-y|^2}\right)\geq\frac{1}{4}\frac{d_xd_y}{|x-y|^2}.\qquad\qquad\square$$

4.3. An estimate for the derivative of the Green function for $-\Delta$ when $n = 2$.

Lemma 4.4. *With Ω a domain in $\mathbb{R}^2$ as above, the Green function on Ω for $-\Delta$ satisfies*

$$|\nabla_x G_\Omega(x, y)| \leq c \frac{1}{|x - y|} \left(1 \wedge \frac{d_y}{|x - y|}\right) \quad \text{for all } x, y \in \Omega. \tag{33}$$

Proof. We skip the subscript Ω. Since $G(\cdot, y)$ is harmonic in $\Omega \backslash \{y\}$, the Poisson formula shows when $x \in B(x_0, R) \subset \Omega$ that

$$G(x, y) = \frac{R^2 - |x - x_0|^2}{2\pi R} \int_{\partial B(x_0, R)} \frac{G(z, y)}{|z - x|^2} \, d\sigma_z \quad \text{for } y \in \Omega \backslash B(x_0, R).$$

We obtain after differentiating and setting $x_0 = x$

$$
\begin{aligned}
\nabla_x G(x, y) &= \frac{R}{\pi} \int_{\partial B(x,R)} \frac{(z - x)}{|z - x|^4} G(z, y) \, d\sigma_z \\
&= \frac{1}{\pi} R^{-3} \int_{\partial B(x,R)} (z - x) G(z, y) \, d\sigma_z. \tag{34}
\end{aligned}
$$

First suppose that $d_x \leq 2|x - y|$. Set $R = \frac{1}{4} d_x$. Let $x^* \in \partial\Omega$ be such that $d_x = |x - x^*|$. Then $d_z \leq |z - x^*| \leq |z - x| + |x - x^*| = 1\frac{1}{4} d_x$ for $z \in \partial B(x, R)$. And it follows from $|x - y| \leq |x - z| + |z - y| = \frac{1}{4} d_x + |z - y| \leq \frac{1}{2}|x - y| + |z - y|$ that $|x - y| \leq 2|z - y|$ for $z \in \partial B(x, R)$.

With (28) and Lemma 4.3 we get

$$G(z, y) \leq c \frac{d_z}{|z - y|} \left(1 \wedge \frac{d_y}{|z - y|}\right) \leq c \frac{\frac{5}{2} d_x}{|x - y|} \left(1 \wedge \frac{2 d_y}{|x - y|}\right) \tag{35}$$

and from (34)

$$
\begin{aligned}
|\nabla_x G(x, y)| &\leq \frac{1}{\pi} R^{-3} 2\pi R \; R \, c \frac{10 R}{|x - y|} \left(1 \wedge \frac{2 d_y}{|x - y|}\right) \\
&\leq 40 \, c \frac{1}{|x - y|} \left(1 \wedge \frac{d_y}{|x - y|}\right). \tag{36}
\end{aligned}
$$

Now suppose that $d_x > 2|x - y|$ and take $R = \frac{1}{2}|x - y|$. Then $\frac{3}{4}d_x \leq d_z \leq \frac{5}{4}d_x$. Since $\int(z - x)\,d\sigma_z = 0$, one has

$$\int_{\partial B(x,R)} (z - x)G(z, y)\,d\sigma_z = \int_{\partial B(x,R)} (z - x)(G(z, y) - G(x, y))\,d\sigma_z. \tag{37}$$

From (28) it follows that for $z \in \partial B(x, R)$

$$G(z, y) - G(x, y) \leq (4\pi)^{-1}\log\left(\frac{1 + c_2\,d_z|z - y|^{-2}d_y}{1 + c_1\,d_x|x - y|^{-2}d_y}\right) \leq c_3, \tag{38}$$

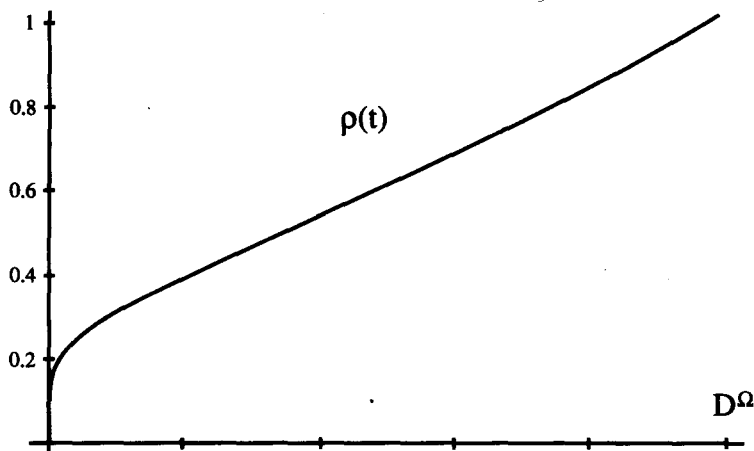where $c_3 = (4\pi)^{-1}\log(5c_2/c_1)$. Similarly,

$$G(z, y) - G(x, y) \geq -(4\pi)^{-1}\log(c_2/c_1). \tag{39}$$

Using (34) we find

$$|\nabla_x G(x, y)| \leq \frac{1}{\pi}R^{-3}2\pi R\,R\,c_3 = 4\,c_3|x - y|^{-1}. \tag{40}$$

Inequality (33) follows since $d_y \geq d_x - |x - y| \geq |x - y|$. $\quad\square$

4.4. 3G type estimates when $n = 2$. Define $\rho(t) = (\log(e\,D^\Omega\,t^{-1}))^{-1}$ and $\Omega_1 = \{z \in \Omega; |x - z| \leq |y - z|\}$.

Lemma 4.5. *If the functions* $G_i(\cdot, \cdot), i = 1, 2,$ *or* 3 *, satisfy* (28), *then there is* $M > 0$ *such that for* $\Xi(x, y, z) = G_1(x, z)G_2(z, y)/G_3(x, y)$ *the following estimates hold. If* $|x - y| > \frac{1}{2}(d_x \vee d_y)$,

$$\Xi(x, y, z) \leq M \, \rho(|z - x|)^{-1} \frac{|x - y|^2}{|z - y|^2} \quad \text{for all } z \in \Omega_1. \tag{41}$$

If $|x - y| < \frac{1}{2}(d_x \vee d_y)$,

$$\Xi(x, y, z) \leq M \, \log\left(1 + \frac{d_x}{|z - x|}\right) \frac{\rho(|x - y|)}{\rho(|z - y|)} \quad \text{for all } z \in \Omega_1. \tag{42}$$

Hence there is M^* *such that for all* $x, y \in \Omega$:

$$\Xi(x, y, z) \leq M^* \frac{\rho(|x - y|)}{\rho(|z - x|)\rho(|z - y|)} \quad \text{for all } z \in \Omega. \tag{43}$$

The last estimate is optimal; for example, for fixed z *and* $\varepsilon < \frac{1}{2}d_z$, *there is* $m > 0$ *such that*

$$\Xi(x, y, z) \geq m \frac{\rho(|x - y|)}{\rho(|z - x|)\rho(|z - y|)} \quad \text{for all } x, y \in \Omega \text{ with } |x - z| \leq \varepsilon$$

$$\text{and } |y - z| \leq \varepsilon. \tag{44}$$

Since $\rho(|z - x|)^{-2}$ is integrable and since it follows from (43) and the inequality $|x - y| \leq 2(|x - z| \vee |y - z|)$ that

$$\Xi(x, y, z) \leq 2 M^*(\rho(|z - x|)^{-1} + \rho(|z - y|)^{-1}) \quad \text{for all } x, y, z \in \Omega, \tag{45}$$

we find:

Corollary 4.6. *With* G_i *as above one finds for* $h \in L^\infty$ *that*

$$\left| \int_\Omega \frac{G_1(x, z)h(z)G_2(z, y)}{G_3(x, y)} \, dz \right| \leq c \, M^* \|h\|_\infty \rho(|x - y|) \quad \text{for all } x \neq y \in \Omega, \tag{46}$$

and for $h \in \mathcal{M}_{2,1}^*$ *that*

$$\left| \int_\Omega \frac{G_1(x, z)h(z)G_2(z, y)}{G_3(x, y)} \, dz \right| \leq c^* \, M^* \|h\|_{2,1}^* \quad \text{for all } x \neq y \in \Omega. \tag{47}$$

Proof. Assume $z \in \Omega_1$. First we consider the case $|x - y| > \frac{1}{2}(d_x \vee d_y)$. If, moreover, $|z - x| \leq d_x$, then we have $d_z \leq d_x + |x - z| \leq 2d_x$. Using Lemma 4.2 and Lemma 4.3 we may estimate by

$$
\begin{aligned}
\Xi(x, y, z) &\leq c \, \log\left(1 + \frac{d_x}{|x - z|}\right) \frac{d_z d_y}{|z - y|^2} \left(\frac{d_x d_y}{|x - y|^2}\right)^{-1} \\
&\leq 2c \, \log\left(1 + \frac{d_x}{|x - z|}\right) \frac{|x - y|^2}{|z - y|^2} \\
&\leq c_1 \, \rho(|z - x|)^{-1} \frac{|x - y|^2}{|z - y|^2}.
\end{aligned}
\tag{48}
$$

If $|z - x| \geq d_x$, then $d_z \leq d_x + |x - z| \leq 2|x - z|$ and

$$
\begin{aligned}
\Xi(x, y, z) &\leq c \frac{d_x}{|x - z|} \frac{d_z d_y}{|z - y|^2} \left(\frac{d_x d_y}{|x - y|^2}\right)^{-1} \\
&\leq c \frac{d_x}{|x - z|} \frac{2|x - z| d_y}{|z - y|^2} \frac{|x - y|^2}{d_x d_y} \\
&\leq 2c \frac{|x - y|^2}{|z - y|^2} \leq 2c \, \rho(|z - x|)^{-1} \frac{|x - y|^2}{|z - y|^2}.
\end{aligned}
\tag{49}
$$

Now assume $|x - y| \leq \frac{1}{2}(d_x \vee d_y)$, and hence that $|x - y| \leq d_x \wedge d_y$. Since $z \in \Omega_1$, we also have $|x - y| \leq |x - z| + |z - y| \leq 2|z - y|$. Using lemma 4.3 and the inequality

$$
\frac{a}{b} \leq \frac{\log(1 + \vartheta a)}{\log(1 + \vartheta b)} \leq \frac{\log(1 + a)}{\log(1 + b)} \quad \text{for } 0 < a < b \text{ and } 0 < \vartheta \leq 1
$$

we find

$$
\begin{aligned}
\Xi(x, y, z) &\leq c \frac{\log(1 + \frac{d_x}{|x-z|}) \log(1 + \frac{d_y}{|y-z|})}{\log(1 + \frac{d_x}{|x-y|}) + \log(1 + \frac{d_y}{|x-y|})} \\
&\leq c \, \log\left(1 + \frac{d_y}{|x - z|}\right) \frac{2 \log(1 + \frac{d_x}{2|y-z|})}{\log(1 + \frac{d_x}{|x-y|})} \\
&\leq 2c \frac{\log(1 + \frac{D^{\Omega}}{2|y-z|})}{\log(1 + \frac{D^{\Omega}}{|x-y|})} \log\left(1 + \frac{d_y}{|x - z|}\right) \\
&\leq 4c \frac{\rho(|x - y|)}{\rho(|z - y|)} \log\left(1 + \frac{d_y}{|x - z|}\right).
\end{aligned}
\tag{50}
$$

To show that the estimate is optimal, one finds by Lemma 4.3 that there is $c > 0$ such that

$$\Xi(x, y, z) \geq c\left(\log\left(2\frac{D^\Omega}{|x-y|}\right)\right)^{-1}\log\left(1+\frac{\varepsilon}{|x-z|}\right)\log\left(1+\frac{\varepsilon}{|y-z|}\right)$$

$$\geq c_\varepsilon\frac{\rho(|x-y|)}{\rho(|z-x|)\rho(|z-y|)}. \qquad \square \qquad (51)$$

Lemma 4.7. *Suppose $G_1(\cdot, \cdot)$ and $G_3(\cdot, \cdot)$ satisfy (28) and $G_2(\cdot, \cdot)$ satisfies (33). Then there is a constant M_2 such that for all $x, y, z \in \Omega$*

$$\frac{G_1(x, z)|\nabla_z G_2(z, y)|}{G_3(x, y)} \leq M_2(\rho(|x-z|)^{-1}|z-y|^{-1} + |x-z|^{-1}). \qquad (52)$$

Since

$$\rho(|x-z|)^{-1}|z-y|^{-1} + |x-y|^{-1} \leq 2 \max\{\rho(|s-z|)^{-1}|s-z|^{-1}; s = x \text{ or } s = y\}$$

we find the following:

Corollary 4.8. *With G_i as above, one has for $k \in \mathcal{M}_{1,1}^*$ that*

$$\int_\Omega \frac{G_1(x, z)|k(z)||\nabla G_2(z, y)|}{G_3(x, y)}\, dz \leq c\, M_2\|k\|_{1,1}^* \quad \text{for all } x \neq y \in \Omega. \qquad (53)$$

Proof. Using (28) and (33) it is sufficient to find a bound for

$$K(x, y, z) = \frac{\log(1 + \frac{d_x d_z}{|x-z|^2})}{\log(1 + \frac{d_x d_y}{|x-y|^2})}|z-y|^{-1}\left(1 \wedge \frac{d_y}{|z-y|}\right). \qquad (54)$$

Again we distinguish two cases.

(i) $|x - y| \leq \frac{1}{2}(d_x \vee d_y)$ and hence $|x - y| \leq d_x \wedge d_y$. Then, with Lemma 4.3,

$$K(x, y, z) \leq 2(\log 2)^{-1}\log\left(1 + \frac{d_x}{|x-z|}\right)|z-y|^{-1}. \qquad (55)$$

(ii) $|x - y| \geq \frac{1}{2}(d_x \vee d_y)$. Using lemma 4.3 again we find

$$K(x, y, z) \leq 4\frac{|x - y|^2}{d_x d_y} \log\left(1 + \frac{d_x d_z}{|x - z|^2}\right) \frac{1}{|z - y|} \left(1 \wedge \frac{d_y}{|z - y|}\right)$$

$$\leq 4\frac{|x - y|^2}{d_x d_y}\left(\frac{d_x}{|x - z|} \wedge \frac{d_x d_z}{|x - z|^2}\right) \frac{1}{|z - y|} \frac{2d_y}{d_y + |z - y|}$$

$$\leq 8\frac{|x - y|^2}{d_x d_y} \frac{d_x}{|x - z|} \frac{d_y}{|z - y|} \left(1 \wedge \frac{d_z}{|x - z|}\right) \frac{1}{d_y + |z - y|}$$

$$\leq 8|x - y|\frac{|x - z| + |z - y|}{|x - z||z - y|}\left(1 \wedge \frac{d_z}{|x - z|}\right) \frac{1}{d_y + |z - y|}$$

$$\leq 8\left(\frac{1}{|x - z|} + \frac{1}{|z - y|}\right)\left(1 \wedge \frac{d_z}{|x - z|}\right) \frac{|x - y|}{d_y + |z - y|}$$

$$\leq 8\left(\frac{1}{|x - z|} + \frac{1}{|z - y|}\right)\left(1 \wedge \frac{d_y + |z - y|}{|x - z|}\right) \frac{|x - z| + |z - y|}{d_y + |z - y|}$$

$$\leq 8\left(\frac{1}{|x - z|} + \frac{1}{|z - y|}\right)\left(\frac{d_y + |z - y|}{|x - z|} \frac{|x - z|}{d_y + |z - y|} + \frac{|z - y|}{d_y + |z - y|}\right)$$

$$\leq 16\left(\frac{1}{|x - z|} + \frac{1}{|z - y|}\right). \tag{56}$$

It follows from (55) and (56) that there is $c \in \mathbb{R}^+$ such that

$$K(x, y, z) \leq c\left(\rho(|x - z|)^{-1}\frac{1}{|z - y|} + \frac{1}{|x - z|}\right). \qquad \square \tag{57}$$

4.5. Other elliptic operators with $n = 2$.

Lemma 4.9.   *With Ω a domain in $\mathbb{R}^2$ as above, the Green function for $-\Delta + C$ with*
$0 \leq C \in C(\overline{\Omega})$ *satisfies* (28) *and* (33).

Proof.   Let $G_c(\cdot, \cdot)$ and $G_0(\cdot, \cdot)$ denote the Green function for $-\Delta + C$, respectively $-\Delta$.
Let $w = (-\Delta + C)^{-1}f$ be the solution of

$$\left[\begin{array}{ll} -\Delta w + C w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{array}\right.$$

Then by the maximum principle one finds for $f > 0$ that

$$(-\Delta)^{-1} f \geq (-\Delta)^{-1}(f - \mathcal{C} \, w) = w. \tag{58}$$

Hence $G_0(x, y) \geq G_c(x, y)$, which shows that G_c satisfies the right-hand side of (28).

Using an argument as for (31) there is a constant $c^\varepsilon > 0$ such that $G_c(x, y) \geq c^\varepsilon \, d_x d_y$ for $|x - y| \geq \varepsilon > 0$, and hence for these x, y the left-hand side of (28) follows. It remains to prove the left-hand side of (28) for $|x - y|$ small. For this we use

$$\begin{aligned} w &= (-\Delta)^{-1}(f - \mathcal{C} \, w) \geq (-\Delta)^{-1}(f - \mathcal{C}(-\Delta)^{-1} f) \\ &= (-\Delta)^{-1} f - (-\Delta)^{-1}(\mathcal{C}(-\Delta)^{-1} f). \end{aligned} \tag{59}$$

which shows that

$$G_c(x, y) \geq G_0(x, y) - \int_\Omega G_0(x, z)\mathcal{C}(z)G_0(z, y) \, dz = G_0(x, y)(1 - H(x, y)), \tag{60}$$

where

$$H(x, y) = (G_0(x, y))^{-1} \int_\Omega G_0(x, z)\mathcal{C}(z)G_0(z, y) \, dz. \tag{61}$$

We will prove for $|x - y|$ small, using (28) and (46) for $G_0(\cdot, \cdot)$ with that c_1 and c_2, respectively cM^*, that

$$G_0(x, y)(1 - H(x, y)) \geq (4\pi)^{-1} \log\left(1 + c_3 \frac{d_x d_y}{|x - y|^2}\right), \tag{62}$$

for $c_3 = \frac{1}{2}c_1 \exp(-2cM^*\|\mathcal{C}\|_\infty \sqrt{c_2})$. The left-hand side of (28) for $G_c(\cdot, \cdot)$ follows from (60) and (62).

From (28) and (46) it follows that

$$\begin{aligned} 4\pi \, G_0(x, y)H(x, y) &\leq cM^*\|\mathcal{C}\|_\infty \log\left(1 + c_2 \frac{d_x d_y}{|x - y|^2}\right) \Big/ \log\left(2\frac{D^\Omega}{|x - y|}\right) \\ &\leq 2cM^*\|\mathcal{C}\|_\infty \log\left(1 + \sqrt{c_2}\frac{D^\Omega}{|x - y|}\right) \Big/ \log\left(1 + \frac{D^\Omega}{|x - y|}\right) \\ &\leq 2cM^*\|\mathcal{C}\|_\infty \sqrt{c_2}. \end{aligned} \tag{63}$$

For $d_x d_y |x - y|^{-2} \geq c_3^{-1}$ we get

$$4\pi \, G_0(x, y) - \log\left(1 + c_3 \frac{d_x d_y}{|x - y|^2}\right)$$

$$\geq \log\left(1 + (c_1 - c_3)\frac{d_x d_y}{|x - y|^2} \Big/ \left(1 + c_3\frac{d_x d_y}{|x - y|^2}\right)\right)$$

$$\geq \log\left(1 + (c_1 - c_3)\, c_3^{-1}\frac{1}{2}\right)$$

$$\geq 2cM^*\|C\|_\infty\sqrt{c_2}, \tag{64}$$

and inequality (62) follows.

Similarly for $d_x d_y |x - y|^{-2} \leq c_3^{-1}$ we have with (28)

$$4\pi \, G_0(x, y) - \log\left(1 + c_3 \frac{d_x d_y}{|x - y|^2}\right) \geq c_4 \frac{d_x d_y}{|x - y|^2}, \tag{65}$$

with $c_4 = c_3^{-1}\log(1 + (c_1 - c_3)c_3^{-1}\frac{1}{2})$.

Now we use

$$4\pi \, G_0(x, y)H(x, y) \leq M_1\|C\|_\infty c_2 \frac{d_x d_y}{|x - y|^2} \Big/ \log\left(2\frac{D^\Omega}{|x - y|}\right). \tag{66}$$

and the fact that we may assume $|x - y| < \varepsilon$, for some arbitrary small $\varepsilon \in \mathbb{R}^+$, to obtain (62).

To show (33) for $G_c(\cdot, \cdot)$ one should notice that

$$G_c(x, y) = G_0(x, y) - \int G_0(x, z)\mathcal{C}(z)G_c(z, y)\, dz \tag{67}$$

and hence

$$\nabla_x G_c(x, y) = \nabla_x G_0(x, y) - \int \nabla_x G_0(x, z)\mathcal{C}(z)G_c(z, y)\, dz. \tag{68}$$

The estimate follows from (33) for $\nabla_x G_0(x, y)$, and from (53) with (28) for the last term. $\qquad\square$

Finally consider elliptic operators L of the following type:

$$L = -\sum_{ij=1}^{2} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} + \left(C + \frac{1}{4}|T^{-2}b|^2\right) \tag{69}$$

where T is the positive (symmetric) matrix such that $T^2 = (a_{ij})$, and $0 \leq C \in C(\overline{\Omega})$. From the

explicit formula

$$G_{L,\Omega}(x, y) = G_{-\Delta + \mathcal{C}(T^{-1}\cdot), T^{-1}\Omega}(T^{-1}x, T^{-1}y)e^{\frac{1}{2}T^{-2}b\cdot(x-y)} \det(T^{-1}).$$ (70)

it follows that

$$(4\pi \ \det(a_{ij}))^{-1} \log\left(1 + c_1\frac{d_x\,d_y}{|x - y|^2}\right) \leq G_{L,\Omega}(x, y)$$

$$\leq (4\pi \ \det(a_{ij}))^{-1} \log\left(1 + c_2\frac{d_x\,d_y}{|x - y|^2}\right),$$ (71)

instead of (28). The Green function $G_{L,\Omega}(\cdot, \cdot)$ will also satisfy the assumptions of Lemmas 4.4, 4.5, and 4.7.

4.6. Proof of Theorem 4.1.
The proof copies the proof of Theorem 3.1 in Section 3.4. Instead of Corollary 3.3 and 3.5 one uses Corollary 4.6 and 4.8.

5. An application to probability theory

Consider the Brownian motion killed on exiting Ω, starting in x, that is conditioned to converge to y. Let $G(\cdot, \cdot)$ be the Green function corresponding with $-\Delta$ on Ω. The expectation for the path lifetime τ_Ω can be expressed by

$$E_y^x\tau_\Omega = \int_\Omega \frac{G(x, z)G(z, y)}{G(x, y)}\,dz$$ (72)

(see [10] or [8]). Cranston and McConnell in [6] and [7] showed

$$E_y^x\tau_\Omega \leq c|\Omega| \quad \text{for } \Omega \subset \mathbb{R}^2$$ (73)

and

$$E_y^x\tau_\Omega \leq c_\Omega \quad \text{for bounded Lipschitz domains in } \mathbb{R}^n.$$ (74)

From Corollaries 3.3 and 4.6 we find, for Ω satisfying the regularity assumptions, that

$$E_y^x\tau_\Omega \leq c_\Omega\left(\log\frac{eD^\Omega}{|x - y|}\right)^{-1} \quad \text{for } x, y \in \Omega \subset \mathbb{R}^2,$$ (75)

$$E_y^x\tau_\Omega \leq c_\Omega|x - y| \quad \text{for } x \in \Omega, \ y \in \partial\Omega \text{ with } \Omega \subset \mathbb{R}^2$$ (76)

$$E_y^x\tau_\Omega \leq c_\Omega|x - y| \quad \text{for } x, y \in \Omega \subset \mathbb{R}^n \text{ with } n \geq 3.$$ (77)

The estimate (75) is optimal. Furthermore, we find that for all $\varepsilon > 0$

$$E_y^x \tau_\Omega \leq c_\varepsilon c_\Omega |x - y|^{2-\varepsilon} \quad \text{for } x, y \in \Omega \subset \mathbb{R}^n \text{ with } n \geq 4. \tag{78}$$

References

[1]   Ancona, A. Comparaison des mesures harmoniques et des fonctions de Green pour des opérateurs elliptiques sur un domaine lipschitzien. *C. R. Acad. Sci. Paris* **294**, 505–508 (1982).

[2]   Bonnet, A. Une propriété liée à la positivité pour l'opérateur biharmonique. Preprint (1989).

[3]   Caristi, G., and Mitidieri, E. Maximum principles for a class of non cooperative elliptic systems. *Delft Progress Report* **14**, 33–56 (1990).

[4]   Caristi, G., and Mitidieri, E. Further results on maximum principles for non cooperative elliptic systems. *Nonl. Ana. T.M.A.* **17**, 547–558 (1991).

[5]   Courant, R. *Dirichlet's Principle, Conformal Mapping and Minimal Surfaces*. New York: Interscience 1950.

[6]   Cranston, M., and McConnell, T. R. The lifetime of conditioned Brownian motion. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **65**, 1–11 (1983).

[7]   Cranston, M. Lifetime of conditioned Brownian motion in Lipschitz domains. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **70**, 335–340 (1985).

[8]   Cranston, M., Fabes, E., and Zhao, Z. Conditional Gauge and potential theory for the Schrödinger operator. *Trans. Amer. Math. Soc.* **307**, 171–194 (1988).

[9]   Cranston, M., and Zhao, Z. Conditional transformation of drift Formula and potential theory for $\frac{1}{2}\Delta + b(\cdot) \cdot \nabla$. *Commun. Math. Phys.* **112**, 613–625 (1987).

[10] De Figueiredo, D., and Mitidieri, E. Maximum principles for linear elliptic systems. *Rend. Ist. Mat. Univ. Trieste* **24** (1992).

[11] Doob, J. L. Conditional Brownian motion and the boundary limits of harmonic functions. *Bull. Soc. Math. France* **85**, 431–458 (1957).

[12] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order*. New York: Springer-Verlag 1977.

[13] Hille, E. *Analytic Function Theory*, Vol. II. Waltham, MA: Blaisdell 1962.

[14] Hueber, H., and Sieveking, M. Uniform bounds for quotients of Green functions on $C^{1,1}$-domains. *Ann. Inst. Fourier* **32**, 105–117 (1982).

[15] Hueber, H. A uniform estimate for Green functions on $C^{1,1}$-domains. Preprint, Universität Bielefeld / Forschungszentrum Bielefeld-Bochum-Stochastik.

[16] Protter, M., and Weinberger, H. *Maximum Principles in Differential Equations*. Prentice Hall 1976.

[17] Simon, B. Schrödinger semigroups. *Bull. Amer. Math. Soc.* **7**, 447–526 (1982).

[18] Sweers, G. A strong maximum principle for a noncooperative elliptic system. *SIAM J. Math. Anal.* **20**, 367–371 (1989).

[19] Sweers, G. Strong positivity in $C(\bar{\Omega})$ for elliptic systems. *Math. Z.* **209**, 251–271 (1992).

[20] Widman, K.-O. Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. *Math. Scand.* **21**, 17–37 (1967).

[21] Zhao, Z. Green function for Schrödinger operator and conditioned Feynman-Kac gauge. *J. Math. Anal. Appl.* **116** (1986), 309–334.

[22] Zhao, Z. Green functions and conditioned gauge theorem for a 2-dimensional domain. In *Seminar on Stochastic Processes, 1987*, edited by E. Çinlar et al., pp. 283–294. Boston: Birkhäuser 1988.

[23] Herbst, I. W., and Zhao, Z. Note on a 3G theorem ($d = 2$). In: *Seminar on Stochastic Processes, 1988*, edited by E. Çinlar et al., pp. 183–184. Boston: Birkäuser 1989.

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