The maximum principle and positive principal eigenfunctions for polyharmonic equations

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1. SECOND ORDER ELLIPTIC EQUATIONS

It is well known that in solving second order elliptic boundary value problems such as

$$\begin{cases} -\Delta u = \lambda u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for quite general domains $\Omega \subset \mathbb{R}^n$ and functions f, one has the following sign results.

- *There is* λ_1 *such that for all* $0 \neq f \geq 0$

1. *if* $\lambda < \lambda_1$, *then there is a solution* u *and* $u > 0$;
2. *if* $\lambda = \lambda_1$, *then there is no solution* u;
3. *if* $\lambda > \lambda_1$, *then* $u \not\geq 0$; *that is, either no solution or if there is a solution there exists* $\tilde{x} \in \Omega$ *such that* $u(\tilde{x}) < 0$.

For bounded smooth domains the number λ_1 is the so-called principal eigenvalue. It has a unique eigenfunction, which is positive, and this eigenfunction is the only positive one (up to normalization).

Two main references for this type of results, which are usually called maximum principles, are the books by Walter (1964) and by Protter and Weinberger (1967). Extensions to general bounded non-smooth domains are studied by Berestycki, Nirenberg and Varadhan (1994).

Due to Clément and Peletier (1979) there is even a property which is called the anti-maximum principle.

- *For every* $0 \neq f \geq 0$ *there is* $\delta_f > 0$ *such that*

4. *if* $\lambda_1 < \lambda < \lambda_1 + \delta_f$, *then there is a solution* u *and* $u < 0$.

A crucial difference with the maximum principle is the fact that the constant δ_f depends on f in general. This result cannot be made uniform in general: that is $\inf\left\{\delta_f; f \in C\left(\bar{\Omega};\mathbb{R}_0^+\right)\right\} = 0$. It is shown (Sweers, 1996b) that, even on smooth domains, $f \in L^p(\Omega)$ with $p > n$ is a necessary condition for the anti-maximum principle to hold. The only uniform result that is known holds for non-Dirichlet boundary conditions in one dimension. See (Clément and Peletier, 1979). The anti-maximum principle is extended to non-smooth domains by Birindelli (1996) but only for right hand sides f that have its support outside of the non-smooth boundary. Behaviour at cone shaped boundary points is studied in (Sweers, 1996a).

Similar results hold for cooperative, weakly coupled systems of second order elliptic equations. For more details see (Walter, 1964), (Mitidieri and De Figueiredo, 1990), (Cosner and Schaefer, 1989), (Pao, 1992), (Mitidieri and Sweers, 1995) and (Sweers, 1992). The last paper also contains an anti-maximum principle for systems.

Systems that are coupled by derivatives or where the coupling is noncooperative in general do not have the sign results mentioned above. Except some higher order operators that can be rewritten as a cooperative system, elliptic operators of order larger than two do not have such features. Remaining positivity preserving properties will be subject of the present paper.

2. POLYHARMONICS ON BALLS

For higher order equations with Dirichlet boundary conditions, such as the polyharmonic, only a very restricted result seems to remain. A basic result goes back 91 years to (Boggio, 1905). Boggio gave an explicit formula for the Green functions of all polyharmonic equations with Dirichlet boundary conditions on the unit ball $B \subset \mathbb{R}^n$ for any n. Dirichlet boundary condition for $(-\Delta)^m$ means that all 0 to $m-1$ derivatives are zero at the boundary. His formula immediately shows one that the Green function is positive:

$$G_{m,n}(x,y) = k_{m,n}\ |x-y|^{2m-n} \int_1^{A(x,y)} \frac{\left(v^2-1\right)^{m-1}}{v^{n-1}} dv, \tag{2.1}$$

with

$$A(x,y) = \frac{[XY]}{|x-y|}.$$

The constants $k_{m,n}$ are positive and the expression $[XY]$ denotes the 'Kelvin-transformed' distance of x and y :

$$[XY] = \left| x\,|y| - \frac{y}{|y|} \right| = \sqrt{|x|^2\,|y|^2 - 2x\cdot y + 1}.$$

Positivity of $G_{m,n}(\cdot,\cdot)$ follows since $[XY] > |x-y|$ implies $A(\cdot,\cdot) > 1$ on B^2. Indeed

$$[XY]^2 - |x-y|^2 = \left(1-|x|^2\right)\left(1-|y|^2\right) > 0 \text{ on } B^2.$$

The solution of

$$\begin{cases} (-\Delta)^m u = f & \text{in } B, \\ \mathcal{D}_m u = 0 \text{ on } \partial B, \end{cases} \tag{2.2}$$

where $\mathcal{D}_m$ is the Dirichlet boundary condition

$$\mathcal{D}_m u = (D^\alpha u)_{\alpha \in \mathbb{N}^n, |\alpha| \leq m-1},$$

is given by

$$u(x) = \int_{y \in B} G_{m,n}(x, y)\ f(y)\ dy.$$

By a rescaling argument one recovers from (2.1) a similar Green function on the half-space $\mathbb{R}^n_+ = \left\{(x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\right\}$. Replacing $[XY]$ in $A(x, y)$ of (2.1) by

$$[XY] = |x - y^*|$$

where $y^* = (y_1, \ldots, y_{n-1}, -y_n)$ one finds a solution of

$$\begin{cases} (-\Delta)^m u = f & \text{in } \mathbb{R}^n_+, \\ \mathcal{D}_m u = 0 \text{ on } \partial \mathbb{R}^n_+, \end{cases} \tag{2.3}$$

for suitable f. Since we still have that $A(x, y) > 1$ for $x, y \in \mathbb{R}^n_+$ this Green function on $\mathbb{R}^n_+$ is positive.

3. OTHER HIGHER ORDER EQUATIONS AND OTHER DOMAINS

Before Boggio showed that the polyharmonic Green function on the ball is positive, he (1901), and also Hadamard (1908a), conjectured that in arbitrary reasonable domains Ω, $f \geq 0$ implies $u \geq 0$. After Hadamard (1908b) showed that such a result does not hold in annuli with small inner radius, the conjecture remained for convex domains.

For $m = n = 2$ there is a physical interpretation. For Ω being the plate, f the load and u the displacement one obtains the so-called *clamped plate equation*:

$$\begin{cases} (-\Delta)^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial \nu} u = 0 \text{ on } \partial \Omega. \end{cases} \tag{3.1}$$

For (3.1) the conjecture can be formulated as:

Pushing upwards implies bending upwards.

3.1 Examples for non-positivity

The conjecture of Boggio and Hadamard proved to be wrong. In 1949 a counter example appeared by Duffin, soon to be followed by Garabedian (1951), see also (Garabedian, 1986,p. 275), Loewner (1953) and Szegö (1953). Garabedian showed that already in nice domains such as an ellipse in $\mathbb{R}^2$ with the ratio of the half axes $\simeq 2$, the Green function for the biharmonic operator Δ^2 changes sign. Hedenmalm (1994) has numerical evidence that for the ratio of the axes between $1.5933\ldots$ and $2.4716\ldots$ the Green function $G(\cdot,\cdot)$ at x,y, with x,y close to the two opposite extreme points, is negative. For ratio larger then $2.4716\ldots$ the sign of $G(x,\cdot)$, with x fixed close to the end of the longer axis, is expected to change at least twice and for growing ratio we expect an oscillatory behaviour. An elementary proof that an eccentric ellipse gives a counter example has recently been published by Shapiro and Tegmark (1994).

A renewed interest in sign properties for the biharmonic started in the seventies. Osher (1973) studied the Green function for the biharmonic in a wedge. In the eighties Coffman (1982) and Coffman and Duffin (1980) studied the Green function for the biharmonic on rectangles and obtained that the Green function has infinitely many sign-changes near a corner. Also Kozlov, Kondrat'ev and Maz'ya (1990) should be mentioned.

Altogether we may conclude that neither arbitrary smoothness, nor uniform convexity or symmetry of domains yields a positive Green function. The question that comes to ones mind is the following.

Is the polyharmonic on the unit ball the only higher order elliptic operator for which the inverse for the Dirichlet problem is sign preserving?

A trivial answer no is obtained by using the same transformation both for the operator and the domain. But it has been shown that the Green function for the Dirichlet problem is positive for a more general class of elliptic operators than the ones obtained by this trivial transformation.

3.2 Examples for positivity

The perturbation of the polyharmonic that has been considered in (Grunau and Sweers, 1996a) adds small lower order derivatives to the operator. Which means that $(-\Delta)^m$ is replaced by

$$L = (-\Delta)^m + \sum_{|\alpha|\leq 2m-1} a_\alpha(\cdot)\ D^\alpha, \tag{3.2}$$

with $\alpha \in \mathbb{R}^n$ a multi-index, $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. We were able to show that for $\|a_\alpha\|_\infty$ sufficiently small the corresponding Green function $G_L(\cdot,\cdot)$ remains positive. The proof uses a power series expansion of the Green operator $\mathcal{G}_L$ in terms of $\mathcal{G}_{(-\Delta)^m}$:

$$\mathcal{G}_L = \mathcal{G}_{(-\Delta)^m}\left(\mathcal{I} + \sum_{k=1}^{\infty}\left(-\sum_{|\alpha|<2m-1}^{\infty} M_{a_\alpha}\, D^\alpha\, \mathcal{G}_{(-\Delta)^m}\right)^k\right),$$

where

$$\left(\mathcal{G}_L f\right)(x) = \int_{y\in\Omega} G_L(x,y)\, f(y)\, dy,$$
$$\left(M_a f\right)(x) = a(x)\, f(x).$$

The crucial step is the pointwise estimate

$$G_{(-\Delta)^m}(x,y) \geq c \left| \int_{z\in B} G_{(-\Delta)^m}(x,z)\, D_z^\alpha G_{(-\Delta)^m}(z,y)\, dz \right| \tag{3.3}$$

(for some $c > 0$) which is a polyharmonic equivalent of the so-called 3G-Theorem of Cranston, Fabes and Zhao (1988). Their theorem holds for bounded Lipschitz domains. Our estimates, valid only on B, are proved by pointwise estimates for the Green function and its derivatives. For these estimates see the appendix.

THEOREM 3.1 (Grunau and Sweers, 1996a) *Let L be defined in* (3.2). *There exists $\varepsilon_{m,n} > 0$ such that if $\|a_\alpha\|_\infty \leq \varepsilon_{m,n}$ for $|\alpha| < 2m$ then for all $f \in L^p(B)$, with B the unit ball in $\mathbb{R}^n$ and $p \in (1,\infty)$, there is a unique solution $u \in W^{2m,p}(B) \cap W_0^{m,p}(B)$ of*

$$\begin{cases} Lu = f & \text{in } B, \\ \mathcal{D}_m u = 0 & \text{on } \partial B. \end{cases} \tag{3.4}$$

Moreover, if $0 \neq f \geq 0$ in Ω then $u(x) > 0$ for all $x \in B$.

Remark 3.1: For $mp > n$ one finds that $u \in C^m\left(\bar{B}\right)$ and for $0 \neq f \geq 0$ an equivalent of Hopf's boundary point Lemma follows for $\|a_\alpha\|_\infty < \varepsilon_{m,n}$; namely $\left(-\frac{\partial}{\partial\nu}\right)^m u > 0$ on ∂B where ν is the outward normal. See (Grunau and Sweers, 1996d).

Domain perturbation yields a more complicated problem. In two dimensions, by using the link with conformal mappings (see Courant, 1950), the following is proven.

THEOREM 3.2 (Grunau and Sweers, 1996c) *Fix $m \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^2$ with $\partial\Omega \in C^{2m,\gamma}$ be such that there exists a mapping $g \in C^{2m}\left(\bar{\Omega};\mathbb{R}^2\right)$ with $\|g - Id\|_{C^{2m}}$ small enough and $g(\bar{\Omega}) = \bar{B}$. Then the Green function for*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \mathcal{D}_m u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

is positive.

The similar question for higher dimensional domains ($n > 2$) remains open.

Hedenmalm (1994) exploited the relation with conformal mappings and studied positivity preserving properties on the disk for the operator $\Delta\, |z|^{-2\alpha}\, \Delta$ with $\alpha > -1$ and Dirichlet boundary conditions.

Using pseudoconformal mappings one can even allow small perturbations in the leading order terms of the polyharmonic equation on the ball in $\mathbb{R}^2$ and still have positivity of the solution operator. This type of result can also be found in (Grunau and Sweers, 1996c).

4. POSITIVE RESOLVENTS FOR THE POLYHARMONIC

We consider the elliptic problem

$$\begin{cases} (-\Delta)^m u = \lambda u + f & \text{in } \Omega, \\ \mathcal{D}_m u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded C^{2m}-smooth domain in $\mathbb{R}^n$. Let us define

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega u\ ((-\Delta)^m u)\ dx}{\int_\Omega u^2\, dx}; u \in W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \right\}, \tag{4.2}$$

which is positive by an integration by parts of $\int_\Omega u\ ((-\Delta)^m u)\ dx$ and a repeated use of the Poincaré-Friedrichs inequality. The number λ_1 is the first eigenvalue and we let ϕ_1 denote a corresponding eigenfunction. Assume that ϕ_1 is unique up to normalization. We want to remark that for $m > 1$ in general one doesn't have uniqueness, or positivity, of the first eigenfunction. Both uniqueness and positivity are lost on annuli with very small inner radius (Coffman, Duffin, Shaffer, 1979).

PROPOSITION 4.1 (Grunau and Sweers, 1996a, Theorem 6.1, Lemma 6.2) *Suppose that for some $\tilde{\lambda} < \lambda_1$ the Green function $G_{\tilde{\lambda}}(x, y)$ for* (4.1) *is positive:*

$$G_{\tilde{\lambda}}(x,y) > 0 \text{ for all } x \neq y \in \Omega.$$

Then for all $\lambda \in \left(\tilde{\lambda}, \lambda_1\right)$ the Green function $G_\lambda(x, y)$ is positive and moreover, the first eigenfunction is of fixed sign.

Remark 4.1: A fixed sign implies uniqueness: if an eigenvalue doesn't have a unique eigenfunction then obviously there exists a sign-changing one.

Proof: Since $(-\Delta)^m : W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \to L^2(\Omega)$ is self-adjoint all eigenvalues are real and the geometric multiplicity equals the algebraic multiplicity. Because of (4.2) we find $\lambda_i \geq \lambda_1$. Let us denote the solution operator of (4.1) for $\lambda \notin \{\lambda_i\}_{i=1}^\infty$ by $\mathcal{G}_\lambda$:

$$(\mathcal{G}_\lambda f)(x) := \int_{y\in\Omega} G_\lambda(x,y) f(y)\, dy.$$

The eigenvalues of $\mathcal{G}_{\tilde{\lambda}}$ we denote by $\{\mu_i\}_{i=1}^\infty$. The eigenvalues $\{\lambda_i\}_{i=1}^\infty$ of (4.1) and $\{\mu_i\}_{i=1}^\infty$ are related through $\mu_i = \left(\lambda_i - \tilde{\lambda}\right)^{-1}$. For $\left|\lambda - \tilde{\lambda}\right| < \nu\left(\mathcal{G}_{\tilde{\lambda}}\right)^{-1}$ the following series converges and we find

$$\mathcal{G}_\lambda = \sum_{k=0}^\infty \left(\lambda - \tilde{\lambda}\right)^k \mathcal{G}_{\tilde{\lambda}}^{k+1}.$$

Since $\mathcal{G}_{\tilde{\lambda}} : C(\bar{\Omega}) \to C(\bar{\Omega})$ is a compact integral operator with a strictly positive kernel a theorem of Jentzsch (1912) (a predecessor of the Krein-Rutman Theorem) implies that the spectral radius $\mu = \nu\left(\mathcal{G}_{\tilde{\lambda}}\right)$ is the largest eigenvalue of $\mathcal{G}_{\tilde{\lambda}}$ and that the corresponding eigenfunction is positive. Hence $\nu\left(\mathcal{G}_{\tilde{\lambda}}\right) = \left(\lambda_1 - \tilde{\lambda}\right)^{-1}$ and the series converges for λ with $\left|\lambda - \tilde{\lambda}\right| < \lambda_1 - \tilde{\lambda}$. For $\lambda \in \left[\tilde{\lambda}, \lambda_1\right)$ we do not only find convergence but also that $\mathcal{G}_\lambda$ is positive. □

COROLLARY 4.2 (Grunau and Sweers, 1996a, Corollary 6.4) *Let λ_1 be the first eigenvalue and assume that $\phi_1(x) > 0$ for all $x \in \Omega$. Then there exists $\lambda_c \in [-\infty, \lambda_1]$ such that for λ, u and f as in* (4.1) *we have*

i) if $\lambda \in (\lambda_1, \infty)$ *then* $0 \neq f \geq 0$ *implies* $u \not\geq 0$ *or no solution* u;
ii) if $\lambda = \lambda_1$ *then* $0 \neq f \geq 0$ *implies no solution* u;
iii) if $\lambda \in (\lambda_c, \lambda_1)$ *then* $0 \neq f \geq 0$ *implies* $u > 0$;
iv) if $\lambda = \lambda_c < \lambda_1$ *then* $0 \neq f \geq 0$ *implies* $u \geq 0$;
v) if $\lambda \in (-\infty, \lambda_c)$ *then* $0 \neq f \geq 0$ *implies* $u \not\leq 0$.

Remark 4.2: For second order elliptic operators $\lambda_c = -\infty$; for higher order elliptic operators one finds that $\lambda_c > -\infty$. For the polyharmonic Dirichlet problem we find that $\lambda_c < 0$ if $\Omega = B$. See respectively the counterexample and Theorem 3.1 in (Grunau and Sweers, 1996a).

Remark 4.3: For $\lambda = \lambda_c$ one finds $f > 0$ implies $u \geq 0$. We expect the positivity preserving property to break down at the boundary first. That is, for $\lambda < \lambda_c$ and $|\lambda - \lambda_c|$ small enough the Green function satisfies $G_\lambda(x, y) > 0$ on Ω^2 except for some x, y near $\partial\Omega \times \partial\Omega$. Some numerical evidence is mentioned by Hedenmalm (1994).

Proof of *i) and ii).* The usual multiplication with the eigenfunction for a weight yields after integrating by parts:

$$0 < \int_\Omega \phi_1 f \, dx = \int_\Omega \phi_1 \left((-\Delta)^m - \lambda\right) u \, dx =$$

$$= \int_\Omega \left((-\Delta)^m - \lambda\right) \phi_1 u \, dx = (\lambda_1 - \lambda) \int_\Omega \phi_1 u \, dx. \tag{4.3}$$

For $\lambda_1 < \lambda$ one finds $u \not\geq 0$ and for $\lambda_1 = \lambda$ one gets a contradiction.

iii). Set $\lambda_c = \inf\{\lambda \in [-\infty, \lambda_1); G_\lambda(x, y) > 0 \text{ for all } x \neq y \in \Omega\}$ if the infimum exists; otherwise set $\lambda_c = \lambda_1$. Proposition 4.1 shows that for all $\lambda \in (\lambda_c, \lambda_1)$ the operator $\mathcal{G}_\lambda$ is positive.

iv). A continuity argument for $\lambda \downarrow \lambda_c$ and *iii)* imply *iv).*

v). With (4.3) one finds a contradiction whenever $0 \neq u \leq 0$. □

5. A POSITIVE PRINCIPAL EIGENFUNCTION

As mentioned before we could not solve in dimensions $n \geq 3$ the question whether or not the resolvent remains positive under small smooth perturbations of the domain. We can show however that such small perturbations do not change the positivity of the first eigenfunction. This result will be the consequence of a lemma for more general elliptic operators. Let w be a smooth, strictly positive function on Ω and let L be a self-adjoint elliptic operator on $W_0^{m,2}(\Omega, wdx)$ as follows

$$L = w(x)^{-1} M^*(x,D) \cdot w(x)\ M(x,D) \tag{5.1}$$

where

$$M(x,D) = \sum_{|\alpha|\le m} b_\alpha(x) D^\alpha \quad \text{and} \quad M^*(x,D) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha b_\alpha(x)$$

and with the functions $b_\beta(\cdot)$, possibly vectorvalued, having appropriate regularity and satisfying for some $c > 0$ and for all $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^n$

$$\sum_{|\alpha|=m} \sum_{|\beta|=m} b_\alpha(x) \cdot b_\beta(x) \xi^{\alpha+\beta} \ge c_e |\xi|^{2m}. \tag{5.2}$$

Condition (5.2) shows that for some $c_0 = C(c_e, \|b_\alpha\|_\infty, \Omega) \in \mathbb{R}$ large enough the operator $L + c_0$ is coercive with constant $c_e' = c_e'(c_e, \inf w, \sup w)$:

$$\int_\Omega u\ (L + c_0)\, u\, wdx \ge c_e' \|u\|_{W_0^{m,2}(\Omega, wdx)}.$$

The first eigenvalue of L is then well defined by

$$\lambda_1 = \inf \left\{ \frac{\int_\Omega (Mu \cdot Mu)\ wdx}{\int_\Omega u^2\, wdx}; u \in W_0^{m,2}(\Omega) \right\}. \tag{5.3}$$

Note that $\lambda_1 \ge C(c_0, c_e)$.

We assume that a corresponding eigenfunction ϕ is normalized by

$$\int_\Omega \phi^2\, dx = 1.$$

LEMMA 5.1 *Let $\gamma \in (0,1)$, let the operators L and $\tilde{L}$ be as in* (5.1) *and assume that $\|b_\alpha\|_{C^{m,\gamma}(\bar{\Omega})}$, $\|\tilde{b}_\alpha\|_{C^{m,\gamma}(\bar{\Omega})}$, $\|w\|_{C^{m,\gamma}(\bar{\Omega})}$, $\|\tilde{w}\|_{C^{m,\gamma}(\bar{\Omega})}$, $\|w^{-1}\|_\infty$ and $\|\tilde{w}^{-1}\|_\infty$ are bounded, say by κ. Suppose that the multiplicity of λ_1, the principal eigenvalue of L, is* 1.
Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\|b_\alpha - \tilde{b}_\alpha\|_{C^{m,\gamma}(\bar{\Omega})} \le \delta \quad \textit{and} \quad \|w - \tilde{w}\|_{C^{m,\gamma}(\bar{\Omega})} \le \delta$$

then the multiplicity of $\tilde{\lambda}_1$ is 1 *and the eigenfunction $\tilde{\phi}_1$ (or $-\tilde{\phi}_1$) satisfies*

$$\left\| \phi_1 - \tilde{\phi}_1 \right\|_{C^{2m}(\bar{\Omega})} \le \varepsilon. \tag{5.4}$$

Proof: For short notation we use $W_w^{m,2} := W^{m,2}(\Omega, wdx)$ and $W^{m,2} := W^{m,2}(\Omega)$. First we will estimate the difference in the eigenvalues. Testing with an auxiliary function in (5.3) one finds that $\lambda_1 \le C(\|b_\alpha\|_\infty, \Omega, \min w, \max w)$ and the analogous result for $\tilde\lambda_1$. Hence we have a uniform bound for $\|\phi_1\|_{W_w^{m,2}}$, $\|\tilde\phi_1\|_{W_{\tilde w}^{m,2}}$ and also for $\|\phi_1\|_{W^{m,2}}$, $\|\tilde\phi_1\|_{W^{m,2}}$, $\|\phi_1\|_{W_{\tilde w}^{m,2}}$ and $\|\tilde\phi_1\|_{W_w^{m,2}}$. Let us denote this bound by $C_1 = C(\kappa, c_e, \Omega)$.

Writing $\mathcal{F}_w^M(u) = \int_\Omega (Mu \cdot Mu)\, wdx$ for $u \in W_0^{m,2}(\Omega)$, we find that

$$\tilde\lambda_1 - \lambda_1 = \frac{\mathcal{F}_{\tilde w}^{\tilde M}(\tilde\phi_1)}{\|\tilde\phi_1\|^2_{L^2_{\tilde w}}} - \frac{\mathcal{F}_w^M(\phi_1)}{\|\phi_1\|^2_{L^2_w}} \le \frac{\mathcal{F}_{\tilde w}^{\tilde M}(\phi_1)}{\|\phi_1\|^2_{L^2_{\tilde w}}} - \frac{\mathcal{F}_w^M(\phi_1)}{\|\phi_1\|^2_{L^2_w}} =$$

$$= \frac{\left(\mathcal{F}_{\tilde w}^{\tilde M}(\phi_1) - \mathcal{F}_w^{\tilde M}(\phi_1)\right)\|\phi_1\|^2_{L^2_w} + \left(\mathcal{F}_w^{\tilde M}(\phi_1) - \mathcal{F}_w^M(\phi_1)\right)\|\phi_1\|^2_{L^2_w} + \mathcal{F}_w^M(\phi_1)\left(\|\phi_1\|^2_{L^2_w} - \|\phi_1\|^2_{L^2_{\tilde w}}\right)}{\|\phi_1\|^2_{L^2_{\tilde w}}\|\phi_1\|^2_{L^2_w}} \le$$

$$\le C_1^2 \frac{\left\|\tilde b_\alpha\right\|^2_{L^\infty}\|w-\tilde w\|_{L^\infty}\|w\|_{L^\infty} + \left\|\tilde b_\alpha - b_\alpha\right\|_{L^\infty}\left\|\tilde b_\alpha + b_\alpha\right\|_{L^\infty}\|w\|^2_{L^\infty} + \|b_\alpha\|^2_{L^\infty}\|w\|_{L^\infty}\|w-\tilde w\|_{L^\infty}}{\|\phi_1\|^2_{L^2_{\tilde w}}\|\phi_1\|^2_{L^2_w}} \le$$

$$\le 2\kappa^5 C_1^2 \left(\|w-\tilde w\|_{L^\infty} + \left\|\tilde b_\alpha - b_\alpha\right\|_{L^\infty}\right). \tag{5.5}$$

By a similar argument one estimates $\lambda_1 - \tilde\lambda_1$ to find with $C^* = 2\kappa^5 C_1^2$ that

$$\left|\tilde\lambda_1 - \lambda_1\right| \le C^*\,\delta. \tag{5.6}$$

Next we will estimate the L^2_w−difference of the eigenfunctions. Let P_1 denote the w-weighted projection on ϕ_1 :

$$P_1(u)(x) = \frac{\int_\Omega \phi_1 u\, wdx}{\|\phi_1\|^2_{L^2_w}}\,\phi_1(x).$$

We have for $u \in W_0^{m,2}(\Omega)$ that

$$\mathcal{F}_w^M((I-P_1)u) \ge \lambda_2 \|(I-P_1)u\|^2_{L^2_w}.$$

Then it follows that

$$\mathcal{F}_w^M(u) = \int_\Omega (MP_1u + M(I-P_1)u)^2\, wdx \ge \lambda_1\|P_1u\|^2_{L^2_w} + \lambda_2\|(I-P_1)u\|^2_{L^2_w}$$

implying

$$\mathcal{F}_w^M(\tilde\phi_1) \ge \lambda_1\left\|P_1\tilde\phi_1\right\|^2_{L^2_w} + \lambda_2\left\|(I-P_1)\tilde\phi_1\right\|^2_{L^2_w}. \tag{5.7}$$

Using (5.5) and the similar estimate with L and $\tilde L$ interchanged, we find

$$\frac{\mathcal{F}_{\tilde w}^{\tilde M}(\phi_1)}{\|\phi_1\|^2_{L^2_{\tilde w}}} \le \frac{\mathcal{F}_w^M(\phi_1)}{\|\phi_1\|^2_{L^2_w}} + C^*\,\delta \quad \text{and} \quad \frac{\mathcal{F}_w^M(\tilde\phi_1)}{\|\tilde\phi_1\|^2_{L^2_w}} \le \frac{\mathcal{F}_{\tilde w}^{\tilde M}(\tilde\phi_1)}{\|\tilde\phi_1\|^2_{L^2_{\tilde w}}} + C^*\,\delta$$

implying that

$$\mathcal{F}_w^M(\tilde\phi_1) \le \left\|\tilde\phi_1\right\|^2_{L^2_w}\left(\tilde\lambda_1 + C^*\,\delta\right) \le \left\|\tilde\phi_1\right\|^2_{L^2_w}(\lambda_1 + 2C^*\delta) =$$

$$= \lambda_1 \left\| P_1 \tilde{\phi}_1 \right\|_{L^2_w}^2 + \lambda_1 \left\| (I - P_1)\, \tilde{\phi}_1 \right\|_{L^2_w}^2 + 2C^* \delta \left\| \tilde{\phi}_1 \right\|_{L^2_w}^2 . \tag{5.8}$$

Set $\delta_0 = \frac{\lambda_2 - \lambda_1}{2C^*}$ which is positive since $\lambda_2 > \lambda_1$. Combining (5.7)-(5.8) it follows for $\delta < \delta_0$ that

$$\left\| (I - P_1)\, \tilde{\phi}_1 \right\|_{L^2_w}^2 \leq \frac{2C^* \delta}{\lambda_2 - \lambda_1} \left\| \tilde{\phi}_1 \right\|_{L^2_w}^2$$

and hence

$$\left\| P_1 \tilde{\phi}_1 \right\|_{L^2_w}^2 \geq \left(1 - \frac{2C^* \delta}{\lambda_2 - \lambda_1} \right) \left\| \tilde{\phi}_1 \right\|_{L^2_w}^2 .$$

One has $\phi_1 = \pm \frac{\|\phi_1\|_{L^2_w}}{\|P_1 \tilde{\phi}_1\|_{L^2_w}} P_1 \tilde{\phi}_1$. Assuming a $+$-sign we first estimate

$$\left\| \frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|\phi_1\|_{L^2_w}} \phi_1 - \tilde{\phi}_1 \right\|_{L^2} =$$

$$= \left\| \left(\frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|P_1 \tilde{\phi}_1\|_{L^2_w}} - 1 \right) P_1 \tilde{\phi}_1 - (I - P_1)\, \tilde{\phi}_1 \right\|_{L^2} \leq$$

$$\leq \frac{\|\tilde{\phi}_1\|_{L^2_w} - \|P_1 \tilde{\phi}_1\|_{L^2_w}}{\|P_1 \tilde{\phi}_1\|_{L^2_w}} \left\| P_1 \tilde{\phi}_1 \right\|_{L^2} + \left\| (I - P_1)\, \tilde{\phi}_1 \right\|_{L^2} \leq$$

$$\leq \left(1 - \sqrt{1 - \tfrac{2C^* \delta}{\lambda_2 - \lambda_1}} \right) \frac{\|P_1 \tilde{\phi}_1\|_{L^2}}{\|P_1 \tilde{\phi}_1\|_{L^2_w}} \left\| \tilde{\phi}_1 \right\|_{L^2_w} + \sqrt{\tfrac{2C^* \delta}{\lambda_2 - \lambda_1}} \frac{\|(I - P_1) \tilde{\phi}_1\|_{L^2}}{\|(I - P_1) \tilde{\phi}_1\|_{L^2_w}} \left\| \tilde{\phi}_1 \right\|_{L^2_w} \leq$$

$$\leq 2\kappa \sqrt{\tfrac{2C^* \delta}{\lambda_2 - \lambda_1}}. \tag{5.9}$$

By this estimate and the normalisation $\|\phi_1\|_{L^2} = \left\| \tilde{\phi}_1 \right\|_{L^2} = 1$, we also have

$$\left\| \phi_1 - \tilde{\phi}_1 \right\|_{L^2} \leq$$

$$\leq \left| 1 - \frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|\phi_1\|_{L^2_w}} \right| \|\phi_1\|_{L^2} + \left\| \frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|\phi_1\|_{L^2_w}} \phi_1 - \tilde{\phi}_1 \right\|_{L^2} \leq$$

$$\leq \left| \left\| \tilde{\phi}_1 \right\|_{L^2} - \left\| \frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|\phi_1\|_{L^2_w}} \phi_1 \right\|_{L^2} \right| + 2\kappa \sqrt{\tfrac{2C^* \delta}{\lambda_2 - \lambda_1}} \leq$$

$$\leq \left\| \tilde{\phi}_1 - \frac{\|\tilde{\phi}_1\|_{L^2_w}}{\|\phi_1\|_{L^2_w}} \phi_1 \right\|_{L^2} + 2\kappa \sqrt{\tfrac{2C^* \delta}{\lambda_2 - \lambda_1}} \leq$$

$$\leq 4\kappa \sqrt{\tfrac{2C^* \delta}{\lambda_2 - \lambda_1}}.$$

The C^{2m}-estimates for $\phi_1 - \tilde{\phi}_1$ follow from the regularity theory (Agmon, Douglis, Nirenberg, 1959) for the boundary value problem

$$\begin{cases} (L + c_0) \left(\phi_1 - \tilde{\phi}_1 \right) = f & \text{in } \Omega, \\ \mathcal{D}_m \left(\phi_1 - \tilde{\phi}_1 \right) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

with $f = \left(\lambda_1 - \tilde{\lambda}_1\right)\phi_1 + \left(\tilde{\lambda}_1 + c_0\right)\left(\phi_1 - \tilde{\phi}_1\right) + \left(\tilde{L} - L\right)\tilde{\phi}_1$. Note that the $C^{m,\gamma}$-bounds on $b_\alpha, \tilde{b}_\alpha, w$, and $\tilde{w}$ will be used in $\left(\tilde{L} - L\right)\tilde{\phi}_1$ as well as for the Schauder-type regularity. □

Let Ω be a domain in $\mathbb{R}^n$, $k \in \mathbb{N}$, $\gamma \in [0,1)$ and $\varepsilon > 0$. We call Ω ε-close in $C^{k,\gamma}$-sense to the unit ball B if there exists a surjective mapping $g : C^{k,\gamma}\left(\bar{B};\bar{\Omega}\right)$ such that

$$\|g - Id\|_{C^{k,\gamma}\left(\bar{B}\right)} \leq \varepsilon. \tag{5.11}$$

THEOREM 5.2 *There is $\varepsilon_{m,n} > 0$ such that if Ω is ε-close in $C^{2m,\gamma}$-sense to B with $\varepsilon < \varepsilon_{m,n}$, then the eigenfunction $\phi_{1,\Omega}$ for the first eigenvalue of*

$$\begin{cases} (-\Delta)^m \phi = \lambda\phi & \text{in } \Omega, \\ \mathcal{D}_m\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.12}$$

is unique (up to normalization) and there exists $c > 0$ such that $\phi_{1,\Omega}(x) \geq c\, d(x)^m$ for all $x \in \Omega$.

Here we denote by $d(x)$ the distance of x to the boundary of $\partial\Omega$:

$$d(x) = \inf_{y \in \partial\Omega} |x - y|. \tag{5.13}$$

Proof: Let $g : \bar{B} \to \bar{\Omega}$ be as in (5.11) and denote the inverse by h. For $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small we find that the inverse h of g exists and satisfies

$$\|h - Id\|_{C^{2m,\gamma}\left(\bar{\Omega}\right)} = \mathcal{O}\left(\|g - Id\|_{C^{2m,\gamma}\left(\bar{B}\right)}\right).$$

For $u \in W_0^m(\Omega)$ define $\tilde{u} \in W_0^m(B)$ by

$$\tilde{u}(x) = u(g(x)).$$

For m even one finds that

$$\int_\Omega \left(\Delta^{\frac{m}{2}} u(y)\right)^2 dy = \int_B \left(\tilde{M}\tilde{u}(x)\right)^2 J_g(x)\, dx$$

with the Jacobian $J_g(\cdot) \in C^{2m-1,\gamma}$ ε-close to 1, and $\tilde{M}$ defined by

$$\left(\tilde{M}(\tilde{u})\right)(x) = \left(\Delta^{\frac{m}{2}} u\right)(g(x)) = A^{\frac{m}{2}}\tilde{u}(x)$$

where

$$A = \sum_{k=1}^n \sum_{l=1}^n \left(\left(\nabla h_k \cdot \nabla h_l\right) \circ g(x)\right) \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l} + \sum_{l=1}^n \left(\left(\Delta h_l\right) \circ g(x)\right) \frac{\partial}{\partial x_l}.$$

For m odd

$$\int_\Omega \left(\nabla\Delta^{\frac{m-1}{2}} u(y)\right)^2 dy = \int_B \left(\tilde{M}\tilde{u}(x)\right)^2 J_g(x)\, dx$$

with $\tilde{M} = \left(\tilde{M}_1, \ldots, \tilde{M}_n\right)$ defined by

$$\left(\tilde{M}_i\left(\tilde{u}\right)\right)(x) = \left(\frac{\partial}{\partial y_i}\Delta^{\frac{m-1}{2}}u\right)(g(x)) = \sum_{p=1}^{n}\frac{\partial h_p}{\partial y_i}\frac{\partial}{\partial x_p}A^{\frac{m-1}{2}}\tilde{u}(x).$$

Using this transformation and $\tilde{w} = J_g$ we find that the eigenvalues and eigenfunctions of

$$\begin{cases} (-\Delta)^m \phi = \lambda\phi & \text{in } \Omega, \\ \mathcal{D}_m\phi = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{M}^*\tilde{w}\tilde{M}\tilde{\phi} = \tilde{\lambda}\tilde{w}\tilde{\phi} & \text{in } B, \\ \mathcal{D}_m\tilde{\phi} = 0 & \text{on } \partial B, \end{cases}$$

are corresponding through

$$\lambda_{i,\Omega} = \tilde{\lambda}_i \qquad \text{and} \qquad \phi_{i,\Omega}\circ g = \tilde{\phi}_i.$$

In the next step one shows that $\tilde{L} = \frac{1}{w}\tilde{M}^*\tilde{w}\tilde{M}$ with $\tilde{w} = J_g$ and $L = (-\Delta)^m$ with $w = 1$ satisfy the conditions of Lemma 5.1 whenever $\|g - Id\|_{C^{2m,\gamma}}$ is sufficiently small. Indeed for m even, writing the m^{th}-order terms of $\tilde{M}$ as in (5.1), we find

$$\sum_{\substack{|\beta|=\frac{1}{2}m \\ \beta\in\mathbb{N}^{n\times n}}}\binom{\frac{1}{2}m}{\beta}\left(\prod_{k,l=1}^{n}\left((\nabla h_k\cdot\nabla h_l)\circ g(x)\right)^{\beta_{kl}}\right)\times$$
$$\times\left(\prod_{l=1}^{n}\left(\frac{\partial}{\partial x_l}\right)^{\sum_{k=1}^{n}\beta_{kl}}\right)\left(\prod_{k=1}^{n}\left(\frac{\partial}{\partial x_k}\right)^{\sum_{l=1}^{n}\beta_{kl}}\right)$$

and

$$\|(\nabla h_k\cdot\nabla h_l)\circ g - \delta_{kl}\|_{C^{m,\gamma}} = \mathcal{O}\left(\|g - Id\|_{C^{m+1,\gamma}}\right).$$

The lower order terms of $\tilde{M}$ each contain at least one derivative of $(\nabla h_k\cdot\nabla h_l)\circ g$ of at least order 1 and at most a derivative of $(\nabla h_k\cdot\nabla h_l)\circ g$ of order $m-1$. For $\beta\in\mathbb{N}^n$ with $|\beta| = m-1$ we have

$$\left\|D^\beta(\nabla h_k\cdot\nabla h_l)\circ g\right\|_{C^{m,\gamma}} = \mathcal{O}\left(\|g - Id\|_{C^{2m,\gamma}}\right).$$

Similar results hold for m odd. We also find that

$$\|J_g - 1\|_{C^{m,\gamma}} = \mathcal{O}\left(\|g - Id\|_{C^{m+1,\gamma}}\right).$$

Since $\phi_{1,B}$ has the property above, namely it is the unique principal eigenfunction satisfying the estimate from below by Proposition 4.1 and Remark 3.1, we are done for $\|g - Id\|_{C^{2m,\gamma}}$ sufficiently small by comparing with Lemma 5.1 the first eigenvalues/functions of

$$\begin{cases} (-\Delta)^m \phi = \lambda\phi & \text{in } B, \\ \mathcal{D}_m\phi = 0 & \text{on } \partial B, \end{cases} \quad \text{and} \quad \begin{cases} \tilde{L}\phi = \lambda\phi & \text{in } B, \\ \mathcal{D}_m\phi = 0 & \text{on } \partial B. \end{cases} \tag{5.14}$$

□

PROPOSITION 5.3 (An anti-maximum principle) *We consider* (4.1)*. Suppose that the principal eigenfunction* ϕ_1 *is unique and that for some* $C > 0$ *one has*

$$\phi_1(x) > C\, d(x)^m \ \text{ for all } x \in \Omega.$$

Then for all $f \in L^p(\Omega)$*, with* $p > \frac{n}{m}$ *and* $p \geq 2$*, and* $f > 0$ *there exists* $\delta_f > 0$ *such that the solution* u *of* (4.1) *satisfies*

$$\begin{array}{lll} vi)\ \text{if } \lambda_1 - \delta_f < \lambda < \lambda_1 & \text{then } u > 0, \\ vii)\ \text{if } \quad \lambda_1 < \lambda < \lambda_1 + \delta_f & \text{then } u < 0. \end{array}$$

See (5.13) *for* $d(x)$*.*

Remark 5.1: By the previous theorem the assumption $\phi_1(x) > c'\, d(x)^m$ holds for $\Omega \subset \mathbb{R}^n$ that is ε-close to a ball with ε sufficiently small.

Proof: The proof uses similar steps as Clément and Peletier (1979). Some steps we can simplify because of the special form of our operator.

We write $L^2(\Omega) = [\![\phi_1]\!] \oplus E$ where $E = \left\{u \in L^2(\Omega);\, \int_\Omega u\, \phi_1\, dx = 0\right\}$. The operator $A : \left(W^{2m}(\Omega) \cap W_0^m(\Omega)\right) \to L^2(\Omega)$ defined by $A = (-\Delta)^m$ is self-adjoint. By assumption λ_1 has a unique eigenfunction ϕ_1 and all other eigenvalues are real. Consequently we have $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. Moreover by using the eigenfunction expansion the operator

$$T_2 = A - \lambda_1 I : \left(W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega) \cap E\right) \to E$$

has a well defined inverse $T_2^{-1} f = \sum_{i=2}^{\infty} (\lambda_i - \lambda_1)^{-1} \langle f, \phi_i \rangle\, \phi_i$ and T_2 is an isomorphism. Since one finds for $f_e \in L^p(\Omega) \cap E$ with $2 \leq p \leq \frac{2n}{n-4m}$ that

$$\left\|(A - \lambda_1 I)^{-1} f_e\right\|_{W^{2m,p}} \leq c\left(\left\|A(A - \lambda_1 I)^{-1} f_e\right\|_{L^p} + \left\|(A - \lambda_1 I)^{-1} f_e\right\|_{L^p}\right) \leq$$

$$\leq c\left(\left\|(A - \lambda_1 I)(A - \lambda_1 I)^{-1} f_e\right\|_{L^p} + (1 + |\lambda_1|)\left\|(A - \lambda_1 I)^{-1} f_e\right\|_{L^p}\right) \leq$$

$$\leq c\,\|f_e\|_{L^p} + c'\left\|(A - \lambda_1 I)^{-1} f_e\right\|_{W^{2m,2}} \leq$$

$$\leq c\,\|f_e\|_{L^p} + c''\,\|f_e\|_{L^2} \leq \left(c + c''\right)\|f_e\|_{L^p}$$

also

$$T_p = A - \lambda_1 I : \left(W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \cap E\right) \to \left(L^p(\Omega) \cap E\right)$$

is an isomorphism for such p. A bootstrapping argument shows that T_p is an isomorphism for all $p \in [2, \infty)$. Note that since $W^{2m,p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$ for $p > \frac{n}{m}$ (see Gilbarg and Trudinger 1983, Theorem 7.26), the boundary conditions are satisfied in the classical sense, implying $(A - \lambda_1 I)^{-1} f_e \in W_0^{m,p}(\Omega)$.

Let $0 < \theta < \lambda_2 - \lambda_1$. Note that $\lambda_2 > \lambda_1$ follows from the assumption and (4.2). Then for $|\lambda - \lambda_1| < \theta$ the operators

$$A - \lambda I : \left(W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \cap E\right) \to \left(L^p(\Omega) \cap E\right)$$

are isomorphisms. For $f_e \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \cap E$ we have

$$\left\| (A - \lambda I)^{-1} f_e \right\|_{W^{2m,p}} =$$

$$= \left\| \sum_{k=0}^{\infty} \left((\lambda - \lambda_1)(A - \lambda_1 I)^{-1} \right)^k (A - \lambda_1 I)^{-1} f_e \right\|_{W^{2m,p}} \leq$$

$$\leq C_\theta \left\| (A - \lambda_1 I)^{-1} f_e \right\|_{W^{2m,p}} \leq C'_\theta \left\| f_e \right\|_{L^p}.$$

Then the solution of $(A - \lambda) u = f$ with $f = c_f \phi_1 + f_e$ and $f_e \in L^p(\Omega) \cap E$ can be written as

$$u = \frac{c_f}{\lambda_1 - \lambda} \phi_1 + (A - \lambda)^{-1} f_e.$$

The continuous imbedding $W^{2m,p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$ for $p > \frac{n}{m}$ shows that

$$\left\| (A - \lambda)^{-1} f_e \right\|_{C^m(\bar{\Omega})} \leq c_{p,m,n,\theta} \left\| f_e \right\|_{L^p}$$

and hence we obtain from the boundary condition that

$$\left| \left((A - \lambda)^{-1} f_e \right)(x) \right| \leq c'_{p,m,n,\theta} \left\| f_e \right\|_{L^p} (d(x))^m.$$

For $f > 0$ it follows that $c_f > 0$ and since $\phi_1(x) > C\, d(x)^m$ we find by

$$u(x) = \frac{c_f}{\lambda_1 - \lambda} \phi_1(x) + \left((A - \lambda)^{-1} f_e \right)(x)$$

$$\begin{cases} \geq \left(\frac{c_f}{\lambda_1 - \lambda} - \frac{c'_{p,m,n,\theta}}{C} \left\| f_e \right\|_{L^p} \right) \phi_1(x) \\ \leq \left(\frac{c_f}{\lambda_1 - \lambda} + \frac{c'_{p,m,n,\theta}}{C} \left\| f_e \right\|_{L^p} \right) \phi_1(x) \end{cases}$$

that for $|\lambda_1 - \lambda| \leq \frac{C\, c_f}{c'_{p,m,n,\theta} \|f_e\|_{L^p}}$ the sign of u equals the sign of $\lambda_1 - \lambda$. □

6. AN APPLICATION TO SEMILINEAR EQUATIONS

The first author (Grunau, 1990 and 1991) studied growth conditions that imply the existence of a strong solution for the following type of problems:

$$\begin{cases} Lu + g(\cdot, u) = f & \text{in } \Omega, \\ \mathcal{D}_m u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

with $L = (-\Delta)^m$ and where $u \mapsto g(\cdot, u)$ exceeds the controllable growth rate $u^{\frac{n+2m}{n-2m}}$. Recently in (Grunau and Sweers, 1996b) results have been extended to the following

$$L = \left(- \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right)^m + \sum_{|\alpha| \leq 2m-1} b_\alpha(x) D^\alpha \tag{6.2}$$

with $a_{ij} \in \mathbb{R}$, $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq c\,|\xi|^2$, $b_\alpha \in C^{|\alpha|,\gamma}(\bar{\Omega})$ and L assumed to be coercive, which means that for some $c > 0$ and all $u \in W_0^{m,2}(\Omega) \cap C^{2m}(\bar{\Omega})$ one has

$$\int_\Omega uLu\,dx \geq c\,\|u\|^2_{W^{m,2}(\Omega)}\,.$$

In this section we will briefly explain the arguments necessary in proving a result as follows. For the sake of argument we assume that g and Ω are sufficiently smooth.

THEOREM 6.1 (Grunau and Sweers, 1996b) *Fix $n \geq 2m$ and suppose that g satisfies the sign condition $ug(\cdot,u) \geq 0$ for all $u \in \mathbb{R}$ and the one-sided growth condition*

$$g(x,u) \geq -c\,(1+|u|^\sigma) \tag{6.3}$$

with

$$\begin{array}{lll} \sigma = 1 & \textit{if} & 6m \leq n, \\ \sigma < \frac{4m}{n-2m} & \textit{if} & 2m < n < 6m, \\ \sigma < \infty & \textit{if} & n = 2m. \end{array} \tag{6.4}$$

Then for every $f \in C^\alpha(\bar{\Omega})$ problem (6.1) has a solution $u \in C^{2m,\alpha}(\Omega) \cap W_0^{m,2}(\Omega)$.

Remark 6.1: We do not consider $n < 2m$ since there is no critical growth rate.

For the following two-sided growth condition, instead of (6.3),

$$\begin{array}{ll} g(x,u) \geq -c\,(1+|u|^\tau) & \text{for } u \leq 0 \\ g(x,u) \leq c\,(1+|u|^\tau) & \text{for } u \geq 0 \end{array} \tag{6.5}$$

with $\tau \leq \frac{n+2m}{n-2m}$ existence of a weak solution $u \in W_0^{m,2}(\Omega)$ follows from the coercivity of (6.1). Moreover, for $\tau < \frac{n+2m}{n-2m}$ a linear argument, bootstrapping between Sobolev imbedding and regularity theory (see (Agmon, Douglis, Nirenberg 1959)), shows existence of a strong solution $u \in C^{2m}(\bar{\Omega})$ as well as regularity of any weak solution. Luckhaus (1979) proved for general elliptic operators that the solutions for (6.1) are classical, meaning $u \in C^{2m}(\bar{\Omega})$, whenever (6.5) holds with $\tau \leq \frac{n+2m}{n-2m}$.

For $m = 1$ no controllable growth conditions are needed. Here the maximum principle together with the sign condition for g give an L^∞-bound to start the bootstrapping. For $m = 2$ Tomi in 1976 obtains a classical solution by using the maximum principle for an auxiliary function like $a\,(\Delta u)^2 + G(u)$ where $G' = g$ and $a \in \mathbb{R}$.

These approaches do not work for general higher order elliptic equations with zero Dirichlet boundary conditions. Not only no maximum principle on general domains exists but also the restriction to a level set defines a new non-zero Dirichlet problem. By exploiting the Green function estimates on balls a local maximum principle can however be proven.

THEOREM 6.2 (A local maximum principle; see (Grunau and Sweers, 1996b)) *Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ be compact, and suppose that L is as in* (6.2). *Let $q \in \mathbb{R}$ be such that $q > \frac{n}{2m}$ and $q \geq 1$. Then there exists $c \in \mathbb{R}$ such that for every $u \in C^{2m}(\bar{\Omega})$, $f \in C^0(\bar{\Omega})$ that satisfy the differential inequality*

$$Lu \leq f \quad in\ \Omega$$

it follows that

$$\sup_{x \in K} u(x) \leq c \left(\left\| f^+ \right\|_{L^q(\Omega)} + \|u\|_{W^{m-1,1}(\Omega)} \right).$$

This local maximum principle is proven by taking the Green function on the unit ball, rescaling it for small balls in Ω and using it for a test function on these balls.

We end with the explanation how the growth rates for $u < 0$ in (6.4) appear. By the coercivity of L and the sign condition for g one finds

$$\|u\|^2_{W^{m,2}(\Omega)} \leq c \int_\Omega uLu\, dx \leq c \int_\Omega u\left(Lu + g(x,u)\right)\, dx \leq c\, \|u\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}$$

implying with Poincaré-Friedrichs that

$$\|u\|_{W^{m,2}(\Omega)} \leq c' \|f\|_{L^2(\Omega)}.$$

We have

$$Lu(x) \leq \left\| f^+ \right\|_{C^0(\bar{\Omega})} - g(x, u(x)) \leq$$

$$\leq \left\| f^+ \right\|_{C^0(\bar{\Omega})} + c\chi_{[u<0]} \left(1 + |u(x)|^\sigma\right).$$

In order to apply the local maximum principle we need the right hand side to be in $L^q(\Omega)$ for some q satisfying $q > \frac{n}{2m}$ and $q \geq 1$. If $2m < n < 6m$ one may take $q = \frac{2n}{\sigma(n-2m)}$, hence $q > \frac{n}{2m} > 1$, to find by the imbedding of $W^{m,2}(\Omega)$ in $L^{\frac{2n}{n-2m}}(\Omega)$ that

$$\|1 + |u|^\sigma\|_{L^q(\Omega)} \leq c\left(1 + \|u\|^\sigma_{L^{\sigma q}(\Omega)}\right) \leq$$

$$\leq c' \left(1 + \|u\|^\sigma_{W^{m,2}(\Omega)}\right) \leq c'' \left(1 + \|f\|^\sigma_{L^2(\Omega)}\right).$$

If $n = 2m$ then any $q > 1$ will do.

For $n \geq 6m$ the number $\frac{4m}{n-2m}$ is less or equal 1. In this case one replaces the operator L by $L + b_0(x)$ for an appropriately chosen function b_0. See (Grunau and Sweers, 1996c). One can show that the dependence of the constant in the local maximum principle on the function b_0 can be controlled.

A. ESTIMATES FOR THE POLYHARMONIC GREEN FUNCTION

DEFINITION A.1 *Let $g, h : \Omega \to \mathbb{R}$ with $g, h \geq 0$. We say that*

$$g(x) \preceq h(x) \text{ on } \Omega$$

if there exists $c > 0$ such that $g(x) \leq c\, h(x)$ for all $x \in \bar{\Omega}$.
We say that

$$g(x) \sim h(x) \text{ on } \Omega$$

if $g(x) \preceq h(x)$ on Ω and $h(x) \preceq g(x)$ on Ω.

Let $G_{m,n}(x, y)$ denote the Green function for

$$\begin{cases} (-\Delta)^m u = f & \text{in } B, \\ D_m u = 0 & \text{on } \partial B. \end{cases}$$

The following estimates are proven in (Grunau and Sweers, 1996a). These estimates are the crucial tools in most of the results mentioned in this paper.

The distance $d(x)$ of $x \in B$ to the boundary satisfies $d(x) = 1 - |x|$.

PROPOSITION A.2 *On B^2 (that is $(x, y) \in B^2$) we have the following.*

1. *For $2m < n$:*
$$G_{m,n}(x, y) \sim |x - y|^{2m-n} \left(1 \wedge \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right).$$
2. *For $2m = n$:*
$$G_{m,n}(x, y) \sim \log\left(1 + \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right).$$
3. *For $2m > n$:*
$$G_{m,n}(x, y) \sim (d(x)\, d(y))^{m - \frac{1}{2}n} \left(1 \wedge \frac{d(x)^{\frac{1}{2}n} d(y)^{\frac{1}{2}n}}{|x - y|^n}\right).$$

PROPOSITION A.3 *Let $\alpha \in \mathbb{N}^n$. Then on B^2 we have the following.*

1. *For $|\alpha| \geq 2m - n$ and n odd, or, $|\alpha| > 2m - n$ and n even:*
   (a) *if $|\alpha| \leq m$ then*
$$|D_x^\alpha G_{m,n}(x, y)| \preceq |x - y|^{2m-n-|\alpha|} \left(1 \wedge \frac{d(x)^{m-|\alpha|} d(y)^m}{|x - y|^{2m-|\alpha|}}\right);$$
   (b) *if $|\alpha| \geq m$ then*
$$|D_x^\alpha G_{m,n}(x, y)| \preceq |x - y|^{2m-n-|\alpha|} \left(1 \wedge \frac{d(y)^m}{|x - y|^m}\right).$$
2. *For $|\alpha| = 2m - n$ and n even:*

(a) *if* $|\alpha| \le m$ *(that is* $m \le n$*) then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \left(1 \wedge \frac{d(x)^{m-|\alpha|} d(y)^m}{|x-y|^{2m-|\alpha|}}\right) ;$$

(b) *if* $|\alpha| \ge m$ *(that is* $m \ge n$*) then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq \log\left(2 + \frac{d(y)}{|x-y|}\right) \left(1 \wedge \frac{d(y)^m}{|x-y|^m}\right).$$

3. *For* $|\alpha| \le 2m - n$ *and* n *odd, or,* $|\alpha| < 2m - n$ *and* n *even:*

(a) *if* $|\alpha| \le m - \frac{1}{2}n$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(x)^{m-\frac{1}{2}n-|\alpha|} d(y)^{m-\frac{1}{2}n} \left(1 \wedge \frac{d(x)^{\frac{1}{2}n} d(y)^{\frac{1}{2}n}}{|x-y|^n}\right);$$

(b) *if* $m - \frac{1}{2}n \le |\alpha| \le m$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(y)^{2m-n-|\alpha|} \left(1 \wedge \frac{d(x)^{m-|\alpha|} d(y)^{n-m+|\alpha|}}{|x-y|^n}\right);$$

(c) *if* $m \le |\alpha|$ *then*

$$|D_x^\alpha G_{m,n}(x,y)| \preceq d(y)^{2m-n-|\alpha|} \left(1 \wedge \frac{d(y)^{n-m+|\alpha|}}{|x-y|^{n-m+|\alpha|}}\right).$$

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