No Gidas-Ni-Nirenberg type result for semilinear biharmonic problems

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1 The problem

A famous result of Gidas, Ni and Nirenberg, [3], following an earlier result of Serrin [5], states that positive solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is a ball in $\mathbb{R}^n$ and f Lipschitz, are radially symmetric and radially decreasing. A similar conjecture (e.g. [2]) has been made for the semilinear biharmonic problem under zero Dirichlet boundary conditions

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B, \\ u = |\nabla u| = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

or with Navier boundary conditions:

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B, \\ u = \Delta u = 0 & \text{on } \partial B. \end{cases} \tag{2}$$

Since the proof of [3] is based on the maximum principle, which is available in its full generality only for (cooperative systems of) second order equations, one would tend to believe that such type of result cannot hold for (1) or (2). Trying to get a negative answer to the question whether or not positive solutions are radially symmetric will necessarily lead to a strict p.d.e. approach and will hence be hard to obtain. The radially decreasing part of the claim however allows an o.d.e.-counterexample as we will show shortly. Let us fix this part in a conjecture:

Conjecture 1 *A radially symmetric positive solution to (1) or (2) is radially decreasing. That is, if $B = \{|x| < R\}$ and $u(x) = u(|x|)$, then $u'(r) \le 0$ for $r \in [0, R]$.*

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Remark 1: Whenever $f(u) \geq 0$ this conjecture holds true for (2). One may write this boundary value problem as a second order system by setting $w = -\Delta u$. Since $w = 0$ on ∂B and $-\Delta w = f(u) \geq 0$ the strong maximum principle implies $w \equiv 0$ or $w > 0$ and hence either $u \equiv 0$ or it holds that u has no interior minimum. Since u is positive it follows that the solution is radially decreasing.

Remark 2: For (1) the boundary conditions do not allow a decoupling as for (2). However it follows from [6, Prop. 1] that if a radial function satisfies $\Delta^2 u \geq 0$ in B and satisfies the zero Dirichlet boundary conditions, this function is radially decreasing. Hence a radially symmetric solution of (1) with $f(u) \geq 0$ is radially decreasing. In dimension 2 an even stronger result holds, see [4], functions u, not necessarily radially symmetric, that satisfy $\Delta^2 u \geq 0$ on a disk B and $u = |\nabla u| = 0$ on ∂B cannot have a local minimum. Since the approach in [4] uses conformal mappings the last result has no direct extension to higher dimensions.

2 The counterexamples

We will restrict ourselves to two dimensions. In $\mathbb{R}^2$ one finds for functions $u(x) = u(|x|)$ that

$$\Delta^2 u = u^{iv}(r) + 2u'''(r)/r - u''(r)/r^2 + u'(r)/r^3.$$

One may check that

$$\begin{aligned} v(r) &= i\left(J_0\left(e^{\frac{1}{4}\pi i} r\right) - I_0\left(e^{\frac{1}{4}\pi i} r\right)\right) = \\ &= i\sum_{k=0}^{\infty}\left(\frac{\left(-\frac{1}{4}e^{\frac{1}{2}\pi i}r^2\right)^k}{(k!)^2} - \frac{\left(\frac{1}{4}e^{\frac{1}{2}\pi i}r^2\right)^k}{(k!)^2}\right) = \sum_{m=0}^{\infty} \frac{2(-1)^m}{((2m+1)!)^2}\left(\tfrac{1}{2}r\right)^{4m+2} \end{aligned}$$

solves $\Delta^2 v + v = 0$ and also that v oscillates with increasing amplitude.

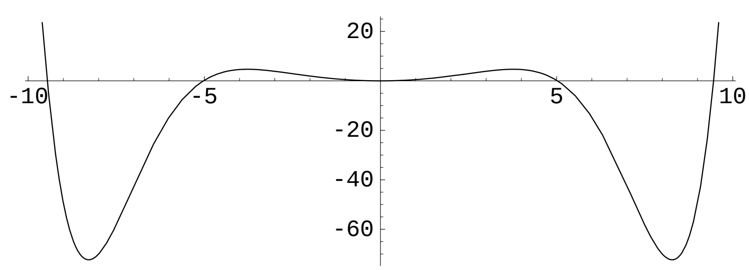

Figure 1: The graph of v.

The functions J_0 and I_0 are the Bessel function and the so-called modified Bessel function of the first kind. See [1].

• In order to obtain a counterexample for (1) let $r_0 > 0$ be the location of the first nonzero minimum of v and set $m_0 = -v(r_0) > 0$. By setting $u_0(r) = v(r) + m_0$ and $f(u) = m_0 - u$ one finds that u_0 is a solution of

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B_{r_0}, \\ u = |\nabla u| = 0 & \text{on } \partial B_{r_0}, \end{cases} \tag{3}$$

which is not radially decreasing. See Figure 2.

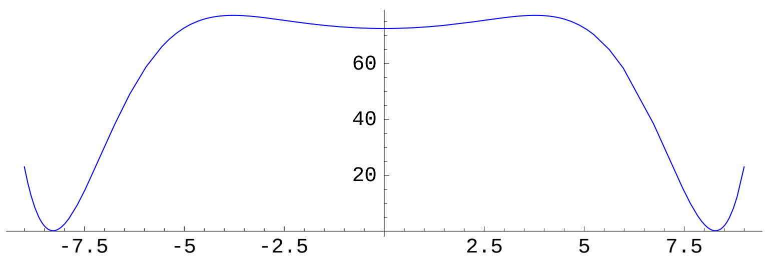

Figure 2: The graph of $u_0 = v + m_0$.

Numerically one finds $r_0 = 8.28099\ldots$ and $m_0 = 72.3308\ldots$.

• For (2) let $r_1 > 0$ be the location of the first zero of $v''(r) + v'(r)/r$ and set $m_1 = -v(r_1) > 0$. By setting $u_1(r) = v(r) + m_1$ and $f(u) = m_1 - u$ one finds that u_1 is a solution of

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B_{r_1}, \\ u = \Delta u = 0 & \text{on } \partial B_{r_1}, \end{cases} \tag{4}$$

which is not radially decreasing.

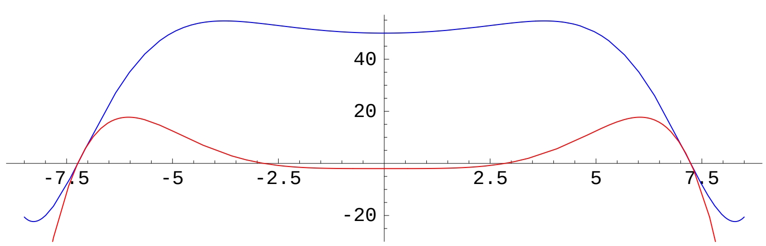

Figure 3: The graphs of $u_1 = v + m_1$ and $-\Delta u_1$ (the lower one in red).

Numerically one finds $r_1 = 7.2388\ldots$ and $m_1 = 50.1554\ldots$.

So we may state:

Proposition 2 *Both (1) and (2) may have positive radially symmetric solutions that are not radially decreasing.*

Remark 3: Although the construction above is in 2 dimensions, the result is not restricted to 2 dimensions. In dimensions $n > 2$ one replaces v by

$$v_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\,\Gamma\big(2m+1+\frac{1}{2}n\big)} \left(\tfrac{1}{2}r\right)^{4m+2}$$

and proceeds in a similar way. This formula even applies in one dimension and $v_1(x) = \sin(\frac{1}{2}\sqrt{2}\,x)\sinh(\frac{1}{2}\sqrt{2}\,x)$ may be used for the construction of a counterexample.

A careful reader might have noticed that we did not use a consequent normalization for the v_n.

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