A NONCOOPERATIVE MIXED PARABOLIC-ELLIPTIC SYSTEM AND POSITIVITY (*)

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Sommario. - *Per quanto concerne la positività, i sistemi cooperativi ellittici e parabolici si comportano come le corrispondenti equazioni: una sorgente positiva implica che la soluzione è positiva. I sistemi con accoppiamento non cooperativo presentano invece un diverso comportamento. Per i sistemi ellittici non cooperativi sussiste un risultato limitato ma uniforme di positività mentre per i sistemi parabolici non cooperativi non esiste alcun risultato di positività. In questo lavoro si esaminano condizioni che assicurino la positività di un sistema intermedio di tipo misto parabolico-ellittico.*

Summary. - *Concerning positivity, cooperative elliptic and parabolic systems behave like the corresponding equations: a positive source implies that the solution is positive. Systems with a noncooperative coupling do not yield such type of behaviour. For noncooperative elliptic systems there is a restricted, but uniform, positivity result and for the noncooperative parabolic system there is no positivity result at all. Here we address positivity preserving properties of an intermediate mixed parabolic-elliptic system.*

The author gratefully acknowledges support from the EU through the project *Reaction-Diffusion Equations* under contract ERBCHRXCT930409.

(*) Pervenuto in Redazione il 1° dicembre 1994.

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1. Introduction.

For the elliptic, respectively parabolic, problem

$$\text{(a)} \begin{cases} -\Delta u = f & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{(b)} \begin{cases} \frac{\partial}{\partial t}u - \Delta u = 0 & \text{in } \Omega \times (0,T)\,, \\ u = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ u\,(0) = f & \text{in } \Omega\ , \end{cases} \tag{1}$$

with Ω a smooth bounded domain in $\mathbb{R}^n$, the classical maximum principle yields that $0 \leq f \in C\,(\bar{\Omega})$ implies $u \geq 0$ (see Protter and Weinberger [17]). Similar results hold for weakly coupled systems of such equations at least when the coupling is cooperative. Let us recall: a system is called weakly coupled if the coupling terms do not contain derivatives; it is called cooperative if the coupling matrix has positive off-diagonal components.

Results for positivity preserving properties of weakly coupled cooperative elliptic and parabolic systems can be found as early as 1964 in [21] by Walter. Also [17] from 1967 contains such a result. In recent years there has been a renewed interest in positivity preserving results for systems. Let us refer to [6], [3], [22], [8], [14] or [16], which were all concerned with the cooperative case or similar to cooperative case. (In a semilinear setting cooperative is replaced by quasimonotone.) Weinberger in [23] addressed the question when a system is similar to cooperative.

On page 192 of [17] cooperative coupling is called and shown to be a genuine restriction for results on systems of the form f positive implies u positive. Indeed, demanding positivity of the solution for all positive source terms implies that the system is cooperative. Nevertheless, one has been able to show restricted positivity, that is, assuming that the source term lies in some subcone of the positive cone, the solution of the non cooperative elliptic system with small non cooperative terms is positive. The main difficulty is to get a uniform result in the sense that the smallness of the non cooperative terms should not depend on the source term.

First results in this direction are found in [5], [18] and [7]. Using the 3G-Theorem of [24], [2], restricted positivity results for more general non cooperative elliptic systems were obtained in [19], [15] and [20]. The last paper also contains positivity results for strongly coupled systems.

The result behind the 3G-Theorem is a uniform estimate of the Green function from below that can be proven (see [24]) using Harnack type estimates. Such uniform estimates do not exist for the kernel functions of the parabolic equation. Hence for a non cooperative parabolic system one cannot expect a result as for the elliptic case.

The motivation of this paper was the question whether a restricted positivity result can be shown for a mixed parabolic-elliptic system. Such a system appears as a limit case in [12].

The mixed parabolic-elliptic system that we consider is as follows

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u &= -\varepsilon v \quad \text{on} \quad \Omega \times (0,\infty)\,, \\ -\Delta v &= u \quad \text{on} \quad \Omega \times (0,\infty)\,, \\ u = v &= 0 \quad \text{on} \quad \partial\Omega \times (0,\infty)\,, \\ u\,(0) &= u_0 \quad \text{on} \quad \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n > 2$ and $\partial\Omega \in C^3$. The question we address is the following:

When does $u_0 \geq 0$ *imply* $u\,(t) \geq 0$ *?*

For $\varepsilon < 0$ the signs of u and v support each other (the coupling is cooperative) and indeed, for $\varepsilon \leq 0$ the system can be shown to be positivity preserving. But for $\varepsilon > 0$ positivity of u yields a positive v and a positive v reduces u. In such a noncooperative system we have to balance the two effects. For closely related fully parabolic and fully elliptic systems the balance is more obvious as we will explain in the next section.

At several places we will use the eigenfunctions and eigenvalues of $-\Delta$ on Ω with Dirichlet boundary condition. They are denoted by $(\lambda_i, \varphi_i)_{i=1}^{\infty}$ with $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. We assume them to be normalized by $\|\varphi_i\|_{L^2(\Omega)} = 1$ and we take $\varphi_1 > 0$.

2. Related parabolic and elliptic systems.

For the parabolic problem

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u &= -\varepsilon v \quad \text{on} \quad \Omega \times (0, \infty), \\ \left(\frac{\partial}{\partial t} - \Delta\right) v &= u \quad \text{on} \quad \Omega \times (0, \infty), \\ u = v &= 0 \quad \text{on} \quad \partial\Omega \times (0, \infty), \\ u(0) &= u_0 \quad \text{on} \quad \Omega, \\ v(0) &= 0 \quad \text{on} \quad \Omega, \end{cases} \tag{3}$$

one directly finds that positivity is not preserved for any $\varepsilon > 0$. For example the function $u(t) = e^{-\lambda_1 t} \cos(\sqrt{\varepsilon} t)\, \varphi_1$ solves (3) for $u_0 = \varphi_1$ and changes sign. Since for large t the solution is dominated by its projection on the first eigenvalue, no non trivial initial value will give a solution that remains positive.

In [15] and [19] it is shown that for the elliptic system

$$\begin{cases} -\Delta u &= f - \varepsilon v \quad \text{on} \quad \Omega, \\ -\Delta v &= u \quad \text{on} \quad \Omega, \\ u = v &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4}$$

there exists $\varepsilon^* > 0$ but small such that for all $\varepsilon \in [0, \varepsilon^*]$ we have $f > 0$ implies that the solution satisfies $u > 0$.

The system in (2) is an intermediate problem. Using the same initial value $u_0 = \varphi_1$ one finds a positive solution for all time t . However, we will show that there is no uniform positivity as for (4). Nevertheless we will show that there exist some positivity preserving properties for (2).

3. Kernel functions.

We start by recalling some classical results. For the elliptic problem (1-a) the solution is given by $u(x) = (\mathcal{G}f)(x)$ with $\mathcal{G}$ defined on $L_2(\Omega)$ by

$$(\mathcal{G}f)(x) = \int_{y\in\Omega} G(x,y)\, f(y)\, dy,$$

and where $G(x,y)$ is the Green function. See [10], [9] or [4]. Since we assumed that Ω is bounded and that $\partial\Omega \in C^3$ there exists c_1 such that

$$\|\mathcal{G}f\|_{W^{2,2}(\Omega)} \le c_1 \;\|f\|_{L^2(\Omega)}.$$

The solution of the parabolic problem (1-b) is given by $u(t,x) = (\mathcal{P}(t)\, f)(x)$, where $\{\mathcal{P}(t)\}_{t\ge 0}$ is an analytic semigroup on $L_2(\Omega)$ defined by

$$(\mathcal{P}(t)\, f)(x) = \int_{y\in\Omega} P(t,x,y)\, f(y)\, dy.$$

The function $P(t,x,y)$ is the standard heat kernel. See [9] or Chapter 4 of [4]. One has

$$\|\mathcal{P}(t)\, f\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\, \|f\|_{L^2(\Omega)} \text{ for all } t > 0,$$

and for some $c_2 > 0$ that

$$\|\mathcal{P}(t)\, f\|_{W^{2,2}(\Omega)} \le c_2\, \frac{1}{t}\; \|f\|_{L^2(\Omega)} \text{ for all } t > 0.$$

Moreover, for all $x \ne y \in \bar{\Omega}$ one has

$$G(x,y) = \int_{t=0}^{\infty} P(t,x,y)\, dt. \tag{5}$$

Inverting the second equation in (2) one obtains

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta + \varepsilon\mathcal{G}\right) u = 0 & \text{on } \Omega\times(0,\infty), \\ u = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(0) = u_0 & \text{on } \Omega. \end{cases} \tag{6}$$

Formally the solution of (6) is $e^{(\Delta-\varepsilon\mathcal{G})t}u_0$. Since $\Delta : W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ generates the C_0-(even analytic) semigroup $\{\mathcal{P}(t)\}_{t\geq 0}$ and since $-\varepsilon\mathcal{G} : L^2(\Omega) \to L^2(\Omega)$ is bounded, $\Delta - \varepsilon\mathcal{G}$ generates a C_0-semigroup on $L^2(\Omega)$, say $\mathcal{U}_\varepsilon(t)$ (Theorem 6.4 of [11]).

We define the operators $\mathcal{S}_\varepsilon(t)$ on $L_2(\Omega)$ by

$$\mathcal{S}_\varepsilon(t) = \mathcal{P}(t) \sum_{k=0}^{\infty} \frac{(-\varepsilon t)^k}{k!} \mathcal{G}^k. \tag{7}$$

Using (5) one finds that the operators $\mathcal{P}(t)$ and $\mathcal{G}$ commute and since $\mathcal{G}$ is bounded, it follows that $\{\mathcal{S}_\varepsilon(t)\}_{t\geq 0}$ is a C_0-semigroup on $L_2(\Omega)$. Moreover, one checks that its generator equals $\Delta - \varepsilon\mathcal{G}$ and hence we find $\mathcal{S}_\varepsilon(t) = \mathcal{U}_\varepsilon(t)$. Defining $u(t) \in L_2(\Omega)$ for $t \geq 0$ by

$$u(t) = \mathcal{S}_\varepsilon(t)\, u_0 \tag{8}$$

one finds that u is the semigroup solution of (6). Note that $\{\mathcal{P}(t)\}_{t\geq 0}$ is an analytic semigroup on $L_2(\Omega)$ with $\|\mathcal{P}(t)\|_{L^2} = e^{-\lambda_1 t}$. Since we also have $\|\mathcal{G}\|_{L^2} = \lambda_1^{-1}$, (7) implies that $\{\mathcal{S}_\varepsilon(t)\}_{t\geq 0}$ is an analytic semigroup on $L_2(\Omega)$ (with the same sector) whenever $|\varepsilon| < \lambda_1^2$.

Instead of $L_2(\Omega)$ we may consider $C(\bar{\Omega})$; $\{\mathcal{P}(t)\}_{t\geq 0}$ is a C_0-semigroup on $C(\bar{\Omega})$ and by the classical maximum principle the operator $-\varepsilon\mathcal{G} : C(\bar{\Omega}) \to C(\bar{\Omega})$ is also bounded. One shows that u, defined in (8), is the classical solution of (6) and hence that $(u, \mathcal{G}u)$ is the classical solution of (2).

Finally we remark that the formula in (7) shows that $S_\varepsilon(t) : C(\bar{\Omega}) \to C(\bar{\Omega})$ is positivity preserving for all $t > 0$ whenever $\varepsilon < 0$.

4. 3-G Theorem.

The following result has been proven by Cranston, Fabes and Zhao in [2] on bounded Lipschitz domains in $\mathbb{R}^n$ with $n \geq 3$. They extended a result of Zhao in [24]. For $n = 2$ see [25], [13] and [20].

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, with $n \geq 2$, whose boundary is $C^{2,\alpha}$. If $G(x,y)$ denotes the Green function for*

$-\Delta$ *with zero Dirichlet boundary condition, then*

$$\operatorname*{ess\,sup}_{x,y\in\Omega} \frac{\int_{z\in\Omega} G(x,z)\, G(z,y)\, dz}{G(x,y)} = M < \infty.$$

This result is used to show the following.

LEMMA 2. *For* $\varepsilon t < 4M^{-1}$ *one has*

$$\mathcal{S}_{\varepsilon}(t) \leq \mathcal{P}(t) \left(\mathcal{I} - \varepsilon t \mathcal{G} + \frac{1}{2} \varepsilon^2 t^2 \mathcal{G}^2 \right).$$

Proof. Note that

$$\left(\left(\mathcal{G} - \gamma \mathcal{G}^2 \right) v \right)(x) =$$

$$= \int_{y\in\Omega} G(x,y)\, v(y)\, dy - \gamma \int_{w\in\Omega} \int_{y\in\Omega} G(x,w)\, G(w,y)\, v(y)\, dy dw =$$

$$= \int_{y\in\Omega} G(x,y) \left(1 - \gamma \frac{\int_{w\in\Omega} G(x,w)\, G(w,y)\, dw}{G(x,y)} \right) v(y)\, dy.$$

Hence for $\varepsilon t < 4M^{-1}$ one finds

$$\mathcal{S}_{\varepsilon}(t) - \mathcal{P}(t) \left(\mathcal{I} - \varepsilon t \mathcal{G} + \frac{1}{2} \varepsilon^2 t^2 \mathcal{G}^2 \right) =$$

$$= -\mathcal{P}(t) \sum_{k=1}^{\infty} \frac{(\varepsilon t)^{2k+1}}{(2k+1)!} \mathcal{G}^{2k} \left(\mathcal{G} - \frac{\varepsilon t}{2k+2} \mathcal{G}^2 \right) \leq$$

$$\leq -\mathcal{P}(t) \sum_{k=1}^{\infty} \frac{(\varepsilon t)^{2k+1}}{(2k+1)!} \mathcal{G}^{2k} \left(\mathcal{G} - \frac{1}{4} \varepsilon t M \mathcal{G} \right) < 0.$$

$\diamondsuit$

If we can show that the operator

$$\mathcal{H}_\varepsilon(t) = \mathcal{P}(t)\left(\mathcal{I} - \varepsilon t \mathcal{G} + \frac{1}{2}\varepsilon^2 t^2 \mathcal{G}^2\right) \tag{9}$$

is not positivity preserving we will have found that $\mathcal{S}_\varepsilon(t)$ is not positivity preserving. Clearly the operator $(\mathcal{I} - \varepsilon t\mathcal{G})$ is not positive and although $\mathcal{P}(t)(\mathcal{I} - \varepsilon t\mathcal{G})$ is also not positive, it might be possible that $\mathcal{H}_\varepsilon(t)$ is. Note that

$$(\mathcal{H}_\varepsilon(t) f)(x) = \int_{y\in\Omega} \Bigg(P(t,x,y) - \varepsilon t \int_{w\in\Omega} P(t,x,w) G(w,y)\, dw + \\ + \tfrac{1}{2}\varepsilon^2 t^2 \int_{w\in\Omega} P(t,x,w) \int_{z\in\Omega} G(w,z) G(z,y)\, dz dw \Bigg) f(y)\, dy.$$

It follows that a necessary and sufficient condition for the positivity of $\mathcal{H}_\varepsilon(t)$, for $t > 0$ but small, is:

$$\left(\int_{w\in\Omega} P(t,x,w) G(w,y)\, dw \right)^2 \leq \\ \leq 2\ P(t,x,y) \left(\int_{w\in\Omega} P(t,x,w) \int_{z\in\Omega} G(w,z) G(z,y)\, dz dw \right), \tag{10}$$

for almost all $(x, y) \in \Omega^2$. By letting $t \to 0$ the left hand side of (10) converges to $(G(x,y))^2$. The right hand side of (10) converges, in distributional sense, for $t \to 0$ to $\delta_y(x) \int_{z\in\Omega} G(x,z) G(z,y)\, dz$ with δ_y the Dirac measure at y. Hence for $x \neq y$ and t small the estimate in (10) cannot be true.

5. (Non) Positivity results.

Lemma 2 and (10) imply that there is no uniform positivity result as for the elliptic system in (4). Nevertheless there are some positivity results. We will show a non positivity result for small t and a positivity result for large t. For the sake of comparison we will state the positivity properties of the parabolic system (3) and the elliptic system (4).

THEOREM 3. (the mixed system) *Let* $u_0 \in C(\bar{\Omega})$. *The function* u *denotes the solution of* (2).

1. *If* $0 \neq u_0 \geq 0$ *and* $u_0(x) = 0$ *for* $x \in B_r(y) \subset \Omega$ *for some* $r > 0$, *then for all* $\varepsilon > 0$ *there is* $t_{\varepsilon,u_0} > 0$ *such that*

$$u(y,t) < 0 \quad \textit{for all } t \in (0, t_{\varepsilon,u_0}).$$

2. *If* $\int_\Omega \varphi_1(x) u_0(x)\, dx > 0$ *holds, then for all* $\varepsilon \in (0, \lambda_1\lambda_2)$ *there exists* $t^*_{\varepsilon,u_0} > 0$ *such that*

$$u(\cdot,t) > 0 \quad \textit{on } \Omega \textit{ for all } t > t^*_{\varepsilon,u_0}.$$

Assume that there exist $c_2 \geq c_1 > 0$ *such that* $c_1\varphi_1(x) \leq u_0(x) \leq c_2\varphi_1(x)$.

3. *There exists a decreasing continuous function* $\tau : (1,\infty) \to \mathbb{R}^+$ *with*
$\lim_{s\downarrow 1} \tau(s) = \infty$ *and* $\lim_{s\to\infty} \tau(s) = 0$ *such that for all* $\varepsilon \in (0, \lambda_1\lambda_2)$:

$$u(\cdot,t) > 0 \quad \textit{on } \Omega \textit{ for all } 0 \leq t < \varepsilon^{-1}\tau\left(\tfrac{c_2}{c_1}\right).$$

4. *There exists an increasing continuous function* $\mathcal{T} : [1,\infty) \to \mathbb{R}^+$ *with* $\mathcal{T}(1) < 0$ *and* $\lim_{s\to\infty} \mathcal{T}(s) = \infty$ *such that for all* $\varepsilon \in (0, \lambda_1\lambda_2)$:

$$u(\cdot,t) > 0 \quad \textit{on } \Omega \textit{ for all } t > \max\left((\lambda_1\lambda_2 - \varepsilon)^{-1}\mathcal{T}\left(\tfrac{c_2}{c_1}\right), 1\right).$$

5. *As a consequence of the last two statements it follows that for all* $\varepsilon \in (0, \lambda_1\lambda_2)$ *there exists* $\kappa_\varepsilon > 1$ *such that if* $\frac{c_2}{c_1} < \kappa_\varepsilon$ *then*

$$u(\cdot,t) > 0 \quad \textit{on } \Omega \textit{ for all } t \geq 0.$$

THEOREM 4 (the parabolic system). *Let* $u_0 \in C(\bar{\Omega})$ *with* $0 \neq u_0 \geq 0$. *The function* u_p *denotes the solution of* (3). *Then*

$$\operatorname{sign}(u_p(\cdot,t)) = \operatorname{sign}(\cos(\sqrt{\varepsilon}\, t)) \qquad \text{on } \Omega \text{ for } t \geq 0.$$

Theorem 5 (the elliptic system). *Let $f \in C(\bar{\Omega})$ with $0 \neq f \geq 0$. The function u_e denotes the solution of* (4)*. Then there is $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*)$:*

$$u_e(\cdot) > 0 \qquad \text{on } \Omega.$$

For a proof of Theorem 5 see [15] or [20]. The proof of Theorem 4 is straightforward. The claim follows since $\mathcal{P}(t)$ is a positivity preserving operator for all $t \geq 0$ and since the solution of (3) satisfies

$$u_p(\cdot, t) = \cos\left(\sqrt{\varepsilon} t\right) \mathcal{P}(t)\, u_0.$$

Remark 1. Similar results as in Theorem 3 hold for the non homogeneous system related to (2) where the first line is replaced by $\left(\frac{\partial}{\partial t} - \Delta\right) u = f - \varepsilon v$. Positivity of f does not imply positivity of u. This can be shown by using the following (formal) expression for the solution u:

$$u(t) = \int_{s=0}^{t} \mathcal{S}_\varepsilon(t-s)\, f(s)\, ds + \mathcal{S}_\varepsilon(t)\, u_0.$$

Remark 2. The theorem for the mixed system can be extended to more general elliptic operators. The two Laplacians that appear in the system may even be replaced by different elliptic operators. In that case the operators $\mathcal{P}(t)$ and $\mathcal{G}$ in general do not commute and the proof will become much more technical.

Proof of Theorem 3. The first claim follows from the argument for (10) that shows that $\mathcal{H}_\varepsilon(t)$ is not positivity preserving. Indeed, assuming $\varepsilon t \leq M^{-1}$, using the estimate for $\mathcal{H}_\varepsilon(t)$ in Lemma 2 and the 3-G Theorem, we find

$$u(y,t) \leq \left(\mathcal{H}_\varepsilon(t)\, u_0\right)(y) =$$

$$= \left(\mathcal{P}(t)\left(\mathcal{I} - \varepsilon t \mathcal{G} + \frac{1}{2}\varepsilon^2 t^2 \mathcal{G}^2\right) u_0\right)(y) \leq$$

$$\leq \left(\mathcal{P}\left(t\right)\left(\mathcal{I}-\frac{1}{2}\varepsilon t\mathcal{G}\right)u_0\right)\left(y\right). \tag{11}$$

Let K be a compact subset of Ω with $y \in K^{\circ}$ and support $(u_0) \cap K \neq \emptyset$. For $x \in K$ one finds by standard estimates (see e.g. [9]) that there are $c_a, c_b > 0$ such that for all $t \in (0,T)$ and $x \in K$, with T some fixed positive number,

$$c_b\, t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}} \leq P\left(t,x,y\right) \leq c_a\, t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}. \tag{12}$$

The estimate on the right hand side of (12) holds for all $x, y \in \Omega$. Moreover, for $w \in K$ one has

$$G\left(w,x\right) \geq c_3 \left|w-x\right|^{2-n}.$$

Let s be such that $B_s\left(y\right) \subset K$. We find for small t:

$$t \int_{w\in\Omega} P\left(t,y,w\right) G\left(w,x\right) dw \geq$$

$$\geq c\ t \int_{|w-y|\leq s} t^{-\frac{n}{2}} e^{-\frac{|y-w|^2}{4t}} \left|w-x\right|^{2-n} dw \geq$$

$$\geq c\ t \int_{|v|\leq \frac{s}{\sqrt{t}}} e^{-\frac{1}{4}|v|^2} \left|y-x+\sqrt{t}v\right|^{2-n} dv \geq c'\ t \left(\frac{1}{|x-y|+s}\right)^{2-n}, \tag{13}$$

and indeed, there exists $c_{u_0,y} > 0$ such that

$$\frac{1}{2}\varepsilon t \left(\mathcal{P}\left(t\right)\mathcal{G}u_0\right)\left(y\right) \geq c_{u_0,y}\ \varepsilon\ t. \tag{14}$$

The right hand side of (12) implies that

$$\left(\mathcal{P}\left(t\right)u_0\right)\left(y\right) \leq c_a \int_{x\in\Omega} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}} u_0\left(x\right)\ dx \leq$$

$$\leq c'_{u_0} \int_{|x-y|>r} t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}\ dx =$$

$$= c'_{u_0} \int_{|v|>\frac{r}{\sqrt{t}}} e^{-\frac{1}{4}|v|^2}\ dv \leq c''_{u_0}\ e^{-\frac{r^2}{8t}}, \tag{15}$$

and then, using (14) and (15), for t small we have

$$\left(\mathcal{P}(t)\left(\mathcal{I}-\frac{1}{2}\varepsilon t\mathcal{G}\right)u_0\right)(y)<0.$$

The second claim. For large t we use

$$L^2(\Omega)=[\![\varphi_1]\!]\oplus[\![\varphi_1]\!]^{\perp},$$

where $[\![\varphi_1]\!]=\{\alpha\varphi_1;\alpha\in\mathbb{R}\}$. Let $\pi_1:L^2(\Omega)\to L^2(\Omega)$ denote the projection on the first eigenfunction. We define $\nu_\varepsilon=\lambda_2+\varepsilon\ \lambda_2^{-1}$. Since $\varepsilon\in(0,\lambda_1\lambda_2)$ we have $\nu_\varepsilon=\inf_{i\geq 2}\left(\lambda_i+\varepsilon\ \lambda_i^{-1}\right)<\lambda_2+\lambda_1$. Since $\mathcal{S}_\varepsilon(t)\varphi_i=e^{-\left(\lambda_i+\varepsilon\ \lambda_i^{-1}\right)}\varphi_i$ and since the eigenfunctions are a complete orthonormal system in $L^2(\Omega)$ we also have for all $\psi\in[\![\varphi_1]\!]^{\perp}$ that

$$\|\mathcal{S}_\varepsilon(t)\,\psi\|_{L^2(\Omega)}\leq e^{-\nu_\varepsilon t}\,\|\psi\|_{L^2(\Omega)}.$$

Notice that $\mathcal{G}$ is a bounded operator on $W^{1,\infty}(\Omega)$. Since $\mathcal{P}(t)$, for $t>0$, is a bounded operator from $L^2(\Omega)$ into $C_0(\bar{\Omega})\cap W^{1,\infty}(\Omega)$ we find that $\mathcal{S}_\varepsilon(t)$, for $t>0$, is also one. Indeed, we have

$$\|\mathcal{S}_\varepsilon(1)\,\psi\|_{W^{1,\infty}(\Omega)}\leq c'\,\|\mathcal{P}(1)\,\psi\|_{W^{1,\infty}(\Omega)},$$

with $c'=e^{\lambda_1\lambda_2\|\mathcal{G}\|_{W^{1,\infty}}}$. Denoting by C_{RK} the constant in the Rellich-Kondrachov imbedding (see Theorem 6.2.II of [1]), we obtain

$$\begin{aligned}\|\mathcal{S}_\varepsilon(1)\,\psi\|_{W^{1,\infty}(\Omega)}&\leq c'\,\|\mathcal{P}(1)\,\psi\|_{W^{1,\infty}(\Omega)}\leq\\ &\leq C_{RK}\ c'\,\|\mathcal{P}(1)\,\psi\|_{W^{1+\frac{n}{2},2}(\Omega)}\leq\\ &\leq c^*\ \|\psi\|_{L^2(\Omega)}\,\text{forall}t\in(0,1]\,,\psi\in L_2(\Omega)\,.\end{aligned}$$

The last estimate uses that $\{\mathcal{P}(t)\}_{t=0}^{\infty}$ is an analytic semigroup on $L^2(\Omega)$. The constant c^* does not depend on $\varepsilon\in(0,\lambda_1\lambda_2)$. Hopf's boundary point Lemma for φ_1 implies that there exists $c>0$, depending only on φ_1, such that

$$(\mathcal{S}_\varepsilon(1)\,\psi)(x)\leq c^*\ c\ \|\psi\|_{L^2(\Omega)}\ \varphi_1(x)\quad\text{ for }x\in\Omega.$$

Hence

$$(\mathcal{S}_\varepsilon(1+t)\,\psi)(x)\leq c^*\ c\ \|\mathcal{S}_\varepsilon(t)\,\psi\|_{L^2(\Omega)}\ \varphi_1(x)\leq$$

$$\leq e^{-\nu_\varepsilon t}\ c^*\ c\ \|\psi\|_{L^2(\Omega)}\ \varphi_1(x) \quad \text{for } x \in \Omega.$$

Writing $u_0 = \pi_1 u_0 + (\mathcal{I} - \pi_1) u_0$ we find that for $t > 1$

$$u(x,t) \geq$$

$$\geq \left(e^{-\left(\lambda_1 + \varepsilon\lambda_1^{-1}\right)t} \|\pi_1 u_0\|_{L^2(\Omega)} - e^{-\nu_\varepsilon t}\ e^{\nu_\varepsilon}\ c^* \cdot\right.$$

$$\left.\cdot c\ \|(\mathcal{I} - \pi_1) u_0\|_{L^2(\Omega)}\ \varphi_1(x) \geq\right.$$

$$\geq c_1 e^{-\left(\lambda_1 + \varepsilon\lambda_1^{-1}\right)t}\left(1 - \frac{C\ e^{-\beta_\varepsilon t}\ \|(\mathcal{I} - \pi_1) u_0\|_{L^2(\Omega)}}{\|\pi_1 u_0\|_{L^2(\Omega)}}\right)\ (\pi_1 u_0)(x) \tag{16}$$

with

$$C = e^{\lambda_2 + \lambda_1}\ c^*\ c, \tag{17}$$

$$\beta_\varepsilon = (\lambda_2 - \lambda_1)\left(1 - \frac{\varepsilon}{\lambda_1 \lambda_2}\right). \tag{18}$$

Since $\varepsilon < \lambda_1 \lambda_2$ and $(\pi_1 u_0)(x) > 0$ we obtain that $u(x,t) > 0$ for large t.

The third claim is shown by using the formula for $\mathcal{S}_\varepsilon(t)$. Putting the odd and even terms together one finds

$$\mathcal{P}(t) \sum_{k=0}^{\infty} \frac{(-\varepsilon t)^k}{k!} \mathcal{G}^k u_0 \geq$$

$$\geq e^{-\lambda_1 t}\left(c_1 \sum_{k=0}^{\infty} \frac{\left(\varepsilon t \lambda_1^{-1}\right)^{2k}}{(2k)!} - c_2 \sum_{k=0}^{\infty} \frac{\left(\varepsilon t \lambda_1^{-1}\right)^{2k+1}}{(2k+1)!}\right) \varphi_1 =$$

$$= c_1 e^{-\lambda_1 t} \cosh\left(\varepsilon t \lambda_1^{-1}\right)\left(1 - \frac{c_2}{c_1} \tanh\left(\varepsilon t \lambda_1^{-1}\right)\right) \varphi_1. \tag{19}$$

The last expression is positive if

$$\varepsilon t < \frac{1}{2} \lambda_1 \ln\left(1 + 2\left(\frac{c_2}{c_1} - 1\right)^{-1}\right).$$

The fourth claim uses (16). We have

$$\|\pi_1 u_0\|_{L^2(\Omega)} \geq c_1 \|\varphi_1\|_{L^2(\Omega)} = c_1,$$

and from the fact that the projection is a contraction it follows that

$$\|(\mathcal{I}-\pi_1)\,u_0\|_{L^2(\Omega)} = \left\|(\mathcal{I}-\pi_1)\left(u_0-\frac{1}{2}\,(c_2+c_1)\,\varphi_1\right)\right\|_{L^2(\Omega)} \leq$$

$$\leq \left\|u_0-\frac{1}{2}\,(c_2+c_1)\,\varphi_1\right\|_{L^2(\Omega)} \leq$$

$$\leq \left\|\frac{1}{2}\,(c_2-c_1)\,\varphi_1\right\|_{L^2(\Omega)} \leq \frac{1}{2}\,(c_2-c_1)\,.$$

Since C in (16) does not depend on ε or t, the estimate in (16) yields

$$u\,(x,t) \geq c_1 e^{-\left(\lambda_1+\varepsilon\lambda_1^{-1}\right)t}\left(1-e^{-\beta_\varepsilon t}\;C\;\frac{1}{2}\left(\frac{c_2}{c_1}-1\right)\right)\;(\pi_1 u_0)\,(x) \tag{20}$$

which implies that $u\,(x,t)$ is positive whenever $t>1$ and

$$(\lambda_1\lambda_2-\varepsilon)\,t > \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\ln\left(\frac{1}{2}C\left(\frac{c_2}{c_1}-1\right)\right).$$

$\diamondsuit$

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