POSITIVITY PROPERTIES OF ELLIPTIC BOUNDARY VALUE PROBLEMS OF HIGHER ORDER

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1. THE PROBLEM

In second order elliptic equations, linear as well as nonlinear, maximum and comparison principles have proved to be extremely powerful and efficient devices. So, for a better understanding of higher order elliptic equations (as e.g. the clamped plate equation or quasilinear curvature equations) it is an obvious step to investigate to what extent similar results do exist there.

As the simple polyharmonic function $x \mapsto -|x|^2$ demonstrates, strong maximum principles are obviously false in higher order elliptic equations. But it is reasonable to ask whether in the Dirichlet problem

$$\begin{cases} Lu = f \text{ in } \Omega, \\ \left(-\dfrac{\partial}{\partial \nu}\right)^j u|\partial\Omega = \varphi_j|\partial\Omega \text{ for } j = 0, \dots, m-1, \end{cases} \tag{1.1}$$

positive data yield positive solutions. Here $\Omega \subset \mathbb{R}^n$ is a sufficiently smooth bounded domain with unit outward normal ν.

We only consider operators L, whose principal part is the m-th power of a second order elliptic operator:

$$Lu = \left(-\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\right)^m u + \sum_{|\alpha| \le 2m-1} b_\alpha(x) D^\alpha u. \tag{1.2}$$

The assumptions on the coefficients will be specified when needed. We use the multi-index notation $D^\alpha = \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$, $\alpha \in \mathbb{N}_0^n$. We want to emphasize that the Dirichlet boundary data (1.1.b) prevent us from writing (1.1) as a system of second order elliptic equations, even if L is simply a power of second order elliptic operators. As far as we know nothing is known about positivity for general elliptic operators of order $2m$.

Numerous counterexamples (see e.g. [1-9]) show that there is in general no affirmative answer to our positivity question. For detailed remarks on the context and the history of this problem we refer to [10].

So the appropriate question is as follows: Are there suitable conditions on the operator L, the domain Ω and choices of boundary data to be prescribed homogeneously ($\varphi_j = 0$ for certain j's) such that positive data yield positive solutions?

In Section 2 we briefly quote results from [11] on the right-hand side. In the present note we focus on the role of the boundary conditions, see Sections 3 and 4.

For applications of positivity results to higher order *nonlinear* equations see e.g. [12-15].

2. THE RIGHT-HAND SIDE

In 1905 Boggio [16] studied the prototype problem

$$\begin{cases} (-\Delta)^m u = f \text{ in } B, \\ \left(\frac{\partial}{\partial \nu}\right)^j u|\partial B = 0 \text{ for } j = 0, \ldots, m-1, \end{cases} \tag{2.1}$$

where $B \subset \mathbb{R}^n$ is the unit ball. By calculating explicitly the corresponding Green function $G_{m,n}$ he showed that this problem preserves positivity:

$$0 \not\equiv f \geq 0 \text{ in } B \quad \Rightarrow \quad u > 0 \text{ in } B.$$

Except a related paper by Hedenmalm [17] no further affirmative result in this direction (e.g. for other domains than B or other operators than $(-\Delta)^m$) has been shown up to now.

Basing on Boggio's formula [16, p.126]

$$G_{m,n}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1} v^{1-n}\, dv, \quad x, y \in B; \tag{2.2}$$

$k_{m,n}$ a known constant, the present authors [11, 10] have proven a perturbation result with respect to positivity.

We fix some $p > 1$. For existence and regularity we refer to [18].

Theorem 2.1. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds.
We assume,

i. if $n = 2$: Ω is a $C^{2m,\gamma}$-smooth bounded domain. There exists a surjective mapping $g : B \to \Omega$, $g \in C^{2m}(\overline{B})$ such that $\|g - id\|_{C^{2m}(\overline{B})} \leq \varepsilon_0$, i.e. Ω is close to B. Moreover $\|a_{ij} - \delta_{ij}\|_{C^{2m-1,\gamma}(\overline{\Omega})} \leq \varepsilon_0$, $\|b_\alpha\|_{C^0(\overline{\Omega})} \leq \varepsilon_0$ for $|\alpha| < 2m$, i.e. L is close to $(-\Delta)^m$.

ii. if $n \geq 3$: $\Omega = B$, $a_{ij} = \delta_{ij}$ and $\|b_\alpha\|_{C^0(\overline{B})} \leq \varepsilon_0$ for $|\alpha| < 2m$.

Then for every $f \in L^p(\Omega)$ there exists a solution $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ to the Dirichlet problem (1.1) with $\varphi_0 = \ldots = \varphi_{m-1} = 0$. Moreover, if $0 \not\equiv f \geq 0$ then the solution is also positive: $u > 0$ in Ω.

3. THE BOUNDARY VALUES, I

In this section we are interested in the role of the Dirichlet datum φ_{m-1} of highest order, while the other boundary data are prescribed homogeneously: $\varphi_0 = \ldots = \varphi_{m-2} = 0$. It seems that this is the only choice of boundary values for which positivity results analogous to Theorem 2.1 do exist. In Section 4 we will comment on the role of the Dirichlet datum of lowest order in biharmonic boundary value problems. There we will have only a relatively restricted perturbation result.

Theorem 3.1. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds.
We assume,

i. if $n = 2$: Ω is a $C^{2m+2,\gamma}$-smooth bounded domain. There exists a surjective mapping $g : B \to \Omega$, $g \in C^{2m+2}(\overline{B})$ such that $\|g - id\|_{C^{2m+2}(\overline{B})} \leq \varepsilon_0$. Moreover $\|a_{ij} - \delta_{ij}\|_{C^{2m+1,\gamma}(\overline{\Omega})} \leq \varepsilon_0$, $\|b_\alpha\|_{C^{|\alpha|}(\overline{\Omega})} \leq \varepsilon_0$ for $|\alpha| < 2m$.

ii. if $n \geq 3$: $\Omega = B$, $a_{ij} = \delta_{ij}$ and $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})} \leq \varepsilon_0$ for $|\alpha| < 2m$.

Then for every $\varphi_{m-1} \in C^0(\partial\Omega)$ there exists a solution $u \in C^{m-1}(\overline{\Omega}) \cap W^{2m,p}_{\text{loc}}(\Omega)$, $p > 1$ arbitrary, to the Dirichlet problem (1.1) with $f = 0$, $\varphi_0 = \ldots = \varphi_{m-2} = 0$. Moreover, if $0 \not\equiv \varphi_{m-1} \geq 0$ then the solution is positive: $u > 0$ in Ω.

Proof. For existence and regularity of the solution we refer to the maximum estimates of Agmon [19] and to the L^p-theory in [18].

First we consider the case $\Omega = B \subset \mathbb{R}^n$, $n \geq 2$, $a_{ij} = \delta_{ij}$. Due to the assumption that $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})}$, $|\alpha| < 2m$, be sufficiently small, we know that the Green function $G_{L,n}(x,y)$ of the Dirichlet problem $Lu = f$ in B, $\left(\frac{\partial}{\partial\nu}\right)^j u|\partial B = 0$ for $j = 0, \ldots, m-1$ exists and is sufficiently smooth with respect to y. If $\varphi_0 = \ldots = \varphi_{m-2} = 0$, then the Poisson kernel corresponding to φ_{m-1} is as follows.

$$\begin{cases} \Delta_y^{m/2} G_{L,n}(x,y), & \text{if } m \text{ is even,} \\ -\dfrac{\partial}{\partial\nu_y}\Delta_y^{(m-1)/2} G_{L,n}(x,y), & \text{if } m \text{ is odd,} \end{cases} \tag{3.1}$$

$x \in B$, $y \in \partial B$. Moreover from [11] we know that $G_{L,n}$ behaves like the Green function $G_{m,n}$ of $(-\Delta)^m$, if ε_0 is small enough:

$$\frac{1}{C} G_{m,n}(x,y) \leq G_{L,n}(x,y) \leq C G_{m,n}(x,y), \quad x, y \in B.$$

The Green function $G_{m,n}$ is positive [16] and vanishes on ∂B precisely of order m. That means for fixed $x \in B$ and $y \to \partial B$ one has:

$$\frac{1}{C_x}(1 - |y|)^m \leq G_{m,n}(x,y) \leq C_x (1 - |y|)^m.$$

With Taylor's formula and the homogeneous boundary conditions of $G_{m,n}$ this yields the positivity of the Poisson kernel (3.1) for $x \in B$, $y \in \partial B$.

To show the more general claim in two dimensions $n = 2$ we apply the same procedure as in [10], where the corresponding step from lower order perturbations to highest order perturbations has been carried out for the right-hand side. The essential devices are conformal mappings, their characterization by means of Green functions for the Laplacian and reduction to canonical form. ∎

If $m = 1$ we recover a special version of the well known strong maximum principle. We have also a "dual" result which could be viewed as generalization of the Hopf boundary lemma.

THEOREM 3.2. There exists $\varepsilon_0 = \varepsilon_0(m,n) > 0$ such that the following holds.

We assume that Ω, a_{ij} and b_α satisfy the smoothness and smallness conditions of Theorem 2.1. Let $f \in C^0(\overline{\Omega})$, $\varphi_1 = \ldots = \varphi_{m-1} = 0$ and let $u \in W^{2m,p}(\Omega) \cap C^{2m-1}(\overline{\Omega})$, $p > 1$ arbitrarily large, be the corresponding solution to the Dirichlet problem (1.1).

Then, if $0 \not\equiv f \geq 0$, we not only have that $u > 0$ in Ω, but furthermore, that $\left(-\frac{\partial}{\partial\nu}\right)^m u > 0$ on $\partial\Omega$.

Proof. As above we first assume that $\Omega = B \subset \mathbb{R}^n$, $n \geq 2$, $a_{ij} = \delta_{ij}$. Let $G_{L,n}$ denote the corresponding Green function:

$$u(x) = \int_B G_{L,n}(x,y) f(y)\, dy, \quad x \in \overline{B},$$

$$\left(-\frac{\partial}{\partial \nu}\right)^m u(x) = \int_B \left(-\frac{\partial}{\partial \nu_x}\right)^m G_{L,n}(x,y) f(y)\, dy, \quad x \in \partial B.$$

The continuity of the coefficients b_α ensures the necessary differentiability of $G_{L,n}$ with respect to x. Next we follow the proof of Theorem 3.1: For every $y \in B$, $G_{L,n}(\,.\,,y)$ vanishes precisely of order m on ∂B. Hence

$$\left(-\frac{\partial}{\partial \nu_x}\right)^m G_{L,n}(x,y) > 0 \text{ for } x \in \partial B,\ y \in B.$$

The two dimensional result again follows by the same reasoning as in [10]. ■

4. THE BOUNDARY VALUES, II

As mentioned above positivity results with respect to boundary values other than φ_{m-1} cannot be expected and are even false in general. But still something can be done in fourth order equations with respect to the lowest order datum φ_0.

To begin with we consider the prototype problem (clamped circular plate)

$$\begin{cases} \Delta^2 u = 0 \text{ in } B, \\ u|\partial B = \varphi_0, \qquad \dfrac{\partial u}{\partial \nu}|\partial B = 0, \end{cases} \tag{4.1}$$

where $B \subset \mathbb{R}^n$ again is the unit ball. The corresponding Poisson kernel, which we denote by K_n,

$$u(x) = \int_{\partial B} K_n(x,y) \varphi_0\, d\omega(y), \quad x \in B,$$

can be calculated from (2.2) by $K_n(x,y) = \frac{\partial}{\partial \nu_y} \Delta_y G_{2,n}(x,y)$. An explicit expression for K_n is also given in [20, p. 34]:

$$K_n(x,y) = \frac{1}{2\omega_n} \frac{(1-|x|^2)^2}{|x-y|^{n+2}} \left\{ n(1 - x \cdot y) - (n-2)|x-y|^2 \right\},\ x \in B,\ y \in \partial B,$$

ω_n denotes the surface area of the unit ball. If $n \geq 5$ then K_n changes sign, but if $n \leq 4$ we have the following estimates.

LEMMA 4.1. For $n \leq 4$, $x \in B$, $y \in \partial B$ we have $K_n(x,y) > 0$. Moreover, on $B \times \partial B$ there holds, if $n = 1, 2, 3$:

$$K_n(x,y) \begin{cases} \preceq |x-y|^{-n-1}(1-|x|)^2, \\ \succeq |x-y|^{-n}(1-|x|)^2, \end{cases} \tag{4.2}$$

and if $n = 4$:

$$K_n(x,y) \sim |x-y|^{-6}(1-|x|)^3. \tag{4.3}$$

For $f, g : M \subset \mathbb{R}^k \to \mathbb{R}^+$ we have used the notation:

$$\begin{aligned} f \sim g \quad &\Leftrightarrow \quad \exists C > 0\ \forall x \in M:\ \frac{1}{C} f(x) \leq g(x) \leq C f(x), \\ f \preceq g \quad &\Leftrightarrow \quad \exists C > 0\ \forall x \in M:\ f(x) \leq C g(x). \end{aligned}$$

Proof of Lemma 4.1. If $n \le 3$ the claim follows from $\frac{1}{2}|x-y|^2 \le 1 - x\cdot y = y\cdot(y-x) \le |y-x|$. If $n=4$ we have $K_4(x,y) = \omega_4^{-1}(1-|x|^2)^3|x-y|^{-6}$. ■

We observe that we have a mild degeneracy from below for $n = 1,2,3$, a strong degeneracy from above and below near ∂B for $n=4$ and change of sign for $n \ge 5$. We believe that also in the perturbation results above (right-hand side and Dirichlet datum of highest order) the transition to change of sign occurs via a degeneracy on the boundary.

Here we also want to perturb the prototype (4.1). As can be seen from [11], cf. also the references therein, for this purpose "3-G-type" results are essential. In this direction we have the following.

LEMMA 4.2. Let $n \in \{1,2,3\}$. On $B \times \partial B \times B$ we have:

$$\frac{|D_z^\alpha G_{2,n}(x,z)|\, K_n(z,y)}{K_n(x,y)} \preceq \begin{cases} 1, & \text{if } |\alpha| \le 3-n, \\ |x-z|^{3-n-|\alpha|} + |y-z|^{3-n-|\alpha|}, & \text{if } |\alpha| \ge 4-n. \end{cases} \tag{4.4}$$

Remark. For $n = 1,2,3$, $|\alpha| \le 2$ these estimates are uniformly (in x,y) integrable with respect to $z \in B$. If $n=4$ one obtains estimates for the quotient in (4.4) with $\frac{1}{1-|x|}$ as a factor. Due to this unbounded factor we are not able to prove perturbation results like Theorem 4.3 below for the case $n=4$.

Proof of Lemma 4.2. To simplify notation, set $d(x) := 1-|x|$ for $x \in B$. We refer to the estimates for $G_{m,n}$ and $|D^\alpha G_{m,n}|$ in [11, Proposition 2.3 and 2.4]. Moreover we use the technical Lemma 3.2 of [11]. We observe that $y \in \partial B$ and that in particular $d(z) \le |y-z|$.
The case: $n = 1,3$ and $|\alpha| \le 4-n$, or $n=2$ and $|\alpha| < 4-n = 2$.
Here we use (8) of [11].

$$\begin{aligned}
\frac{|D_z^\alpha G_{2,n}(x,z)|\, K_n(z,y)}{K_n(x,y)} &\preceq \frac{d(x)^{2-\frac{n}{2}} d(z)^{2-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)^{n/2} d(z)^{n/2}}{|x-z|^n}\right\} \frac{d(z)^2}{|z-y|^{n+1}}}{\frac{d(x)^2}{|x-y|^n}} \\
&\preceq d(x)^{-\frac{n}{2}} d(z)^{4-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)^{\frac{n}{2}} d(z)^{\frac{n}{2}}}{|x-z|^n}\right\} |y-z|^{-n-1}\left(|x-z|^n + |y-z|^n\right) \\
&\preceq d(z)^{4-|\alpha|}|y-z|^{-n-1} + d(x)^{-\frac{n}{2}} d(z)^{4-\frac{n}{2}-|\alpha|}\left(\frac{d(x)}{d(z)}\right)^{\frac{n}{2}} |y-z|^{-1} \\
&= d(z)^{4-|\alpha|}|y-z|^{-n-1} + d(z)^{4-n-|\alpha|}|y-z|^{-1} \preceq |y-z|^{3-n-|\alpha|}.
\end{aligned}$$

The case: $n=2$ and $|\alpha| = 4-n = 2$.

$$\begin{aligned}
\frac{|D_z^\alpha G_{2,2}(x,z)|\, K_2(z,y)}{K_2(x,y)} &\preceq \frac{\log\left(2 + \frac{d(x)}{|x-z|}\right) \min\left\{1, \frac{d(x)^2}{|x-z|^2}\right\} \frac{d(z)^2}{|z-y|^3}}{\frac{d(x)^2}{|x-y|^2}} \\
&\preceq d(x)^{-2} d(z)^2 \left(1 + \frac{d(x)}{|x-z|}\right) \min\left\{1, \frac{d(x)^2}{|x-z|^2}\right\} |y-z|^{-3}\left(|x-z|^2 + |y-z|^2\right) \\
&\preceq d(x)^{-2} d(z)^2 \frac{d(x)^2}{|x-z|^2} |y-z|^{-3}|x-z|^2 + d(x)^{-2} d(z)^2 \left(\frac{d(x)}{d(z)}\right)^2 |y-z|^{-1} \\
&\quad + d(x)^{-2} d(z)^2 \frac{d(x)}{|x-z|}\frac{d(x)}{|x-z|} |y-z|^{-3}|x-z|^2
\end{aligned}$$

$$+d(x)^{-2}d(z)^2\frac{d(x)}{|x-z|}\frac{d(x)}{d(z)}|y-z|^{-1}$$

$$\preceq \frac{d(z)^2}{|y-z|^3}+|y-z|^{-1}+|x-z|^{-1}\frac{d(z)}{|y-z|}\preceq |x-z|^{-1}+|y-z|^{-1}.$$

The case: $n=1,2,3$ and $|\alpha|>4-n$.
In particular we have $|\alpha|\geq 2$.

$$\frac{|D_z^\alpha G_{2,n}(x,z)|\,K_n(z,y)}{K_n(x,y)} \preceq \frac{|x-z|^{4-n-|\alpha|}\min\left\{1,\frac{d(x)^2}{|x-z|^2}\right\}\frac{d(z)^2}{|z-y|^{n+1}}}{\frac{d(x)^2}{|x-y|^n}}$$

$$\preceq d(x)^{-2}d(z)^2|x-z|^{4-n-|\alpha|}|y-z|^{-n-1}\min\left\{1,\frac{d(x)^2}{|x-z|^2}\right\}(|x-z|^n+|y-z|^n)$$

$$\preceq d(x)^{-2}d(z)^2|x-z|^{4-|\alpha|}|y-z|^{-n-1}\frac{d(x)^2}{|x-z|^2}$$

$$+d(x)^{-2}d(z)^2|x-z|^{4-n-|\alpha|}|y-z|^{-1}\frac{d(x)^2}{d(z)^2}$$

$$\preceq |x-z|^{2-|\alpha|}|y-z|^{1-n}+|x-z|^{4-n-|\alpha|}|y-z|^{-1}$$

$$\preceq |x-z|^{3-n-|\alpha|}+|y-z|^{3-n-|\alpha|}.$$

■

THEOREM 4.3. Let $n=1,2$ or 3. Then there exists $\varepsilon_0=\varepsilon_0(n)>0$ such that the following holds.
If $\|b_\alpha\|_{C^{|\alpha|}(\overline{B})}\leq\varepsilon_0$ for $|\alpha|\leq 2$, then for every $\varphi_0\in C^1(\partial B)$ the Dirichlet problem

$$\begin{cases} \Delta^2 u+\sum_{|\alpha|\leq 2} b_\alpha(x)D^\alpha u=0 \text{ in } B,\\ u|\partial B=\varphi_0, \qquad -\frac{\partial u}{\partial\nu}|\partial B=0,\end{cases} \tag{4.5}$$

has a solution $u\in W^{4,p}_{\mathrm{loc}}(B)\cap C^1(\overline{B})$, $p>1$ arbitrary. Moreover, if $0\not\equiv\varphi_0\geq 0$, then the solution is positive: $u>0$ in B.

Proof. For existence and regularity we refer to [19] and [18]. First we assume additionally that $\varphi_0\in C^{4,\gamma}(\partial B)$. Then $\mathcal{K}_n\varphi_0(x):=\int_{\partial B}K_n(x,y)\varphi_0(y)\,d\omega(y)$ maps $\mathcal{K}_n: C^{4,\gamma}(\partial B)\to C^{4,\gamma}(\overline{B})\hookrightarrow W^{4,p}(B)$, $p>1$ arbitrary, see [18]. We write $\mathcal{A}:=\sum_{|\alpha|\leq 2}b_\alpha(\,.\,)D^\alpha$. The solution u of (4.5) is given by $u=-\mathcal{G}_{2,n}\mathcal{A}u+\mathcal{K}_n\varphi_0$ or $(\mathcal{I}+\mathcal{G}_{2,n}\mathcal{A})\,u=\mathcal{K}_n\varphi_0$. Here $\mathcal{I}+\mathcal{G}_{2,n}\mathcal{A}$ is a bounded linear operator in $W^{4,p}(B)$, which for sufficiently small ε_0 is invertible. Hence

$$u=(\mathcal{I}+\mathcal{G}_{2,n}\mathcal{A})^{-1}\mathcal{K}_n\varphi_0=\mathcal{K}_n\varphi_0+\sum_{i=1}^\infty(-\mathcal{G}_{2,n}\mathcal{A})^i\,\mathcal{K}_n\varphi_0.$$

For $i\geq 1$ we integrate by parts. As $\mathcal{A}$ is of order ≤ 2 no additional boundary integrals arise. By means of Fubini-Tonelli we obtain for $x\in B$:

$$(-\mathcal{G}_{2,n}\mathcal{A})^i\,\mathcal{K}_n\varphi_0(x)=(-1)^i\int_{z_1\in B}G_{2,n}(x,z_1)\mathcal{A}_{z_1}\int_{z_2\in B}G_{2,n}(z_1,z_2)\cdot\ldots$$

$$\ldots\cdot\mathcal{A}_{z_{i-1}}\int_{z_i\in B}G_{2,n}(z_{i-1},z_i)\mathcal{A}_{z_i}\int_{y\in\partial B}K_n(z_i,y)\varphi_0(y)\,d\omega(y)dz_i\ldots dz_1$$

$$
\begin{aligned}
&= (-1)^i \int_{z_1\in B} \left(\mathcal{A}^*_{z_1} G_{2,n}(x,z_1)\right) \int_{z_2\in B} \left(\mathcal{A}^*_{z_2} G_{2,n}(z_1,z_2)\right) \ldots \\
&\quad \ldots \int_{z_i\in B} \left(\mathcal{A}^*_{z_i} G_{2,n}(z_{i-1},z_i)\right) \int_{y\in\partial B} K_n(z_i,y)\varphi_0(y) d\omega(y) dz_i \ldots dz_1 \\
&= (-1)^i \int_B \cdots \int_B \int_{\partial B} \left(\mathcal{A}^*_{z_1} G_{2,n}(x,z_1)\right) \left(\mathcal{A}^*_{z_2} G_{2,n}(z_1,z_2)\right) \cdot \ldots \\
&\quad \ldots \cdot \left(\mathcal{A}^*_{z_i} G_{2,n}(z_{i-1},z_i)\right) K_n(z_i,y)\varphi_0(y) d\omega(y) d(z_1,\ldots,z_i).
\end{aligned}
$$

Here $\mathcal{A}^* . = \sum_{|\alpha|\le 2}(-1)^{|\alpha|} D^\alpha(b_\alpha\ .\ )$ is the (formally) adjoint operator of the perturbation $\mathcal{A}$. By virtue of Lemma 4.2 we find:

$$
\begin{aligned}
\left|\left(-\mathcal{G}_{2,n}\mathcal{A}\right)^i \mathcal{K}_n\varphi_0(x)\right| \le\ & \int_{\partial B}\int_B \cdots \int_B K_n(x,y) \frac{\left|\mathcal{A}^*_{z_1} G_{2,n}(x,z_1)\right| K_n(z_1,y)}{K_n(x,y)} \\
& \cdot \frac{\left|\mathcal{A}^*_{z_2} G_{2,n}(z_1,z_2)\right| K_n(z_2,y)}{K_n(z_1,y)} \cdot \ldots \\
& \ldots \cdot \frac{\left|\mathcal{A}^*_{z_i} G_{2,n}(z_{i-1},z_i)\right| K_n(z_i,y)}{K_n(z_{i-1},y)} \varphi_0(y)\, d(z_1,\ldots,z_i) d\omega(y) \\
\le\ & (C_0\varepsilon_0)^i \int_{\partial B} K_n(x,y)\varphi_0(y)\, d\omega(y) = (C_0\varepsilon_0)^i \left(\mathcal{K}_n\varphi_0\right)(x).
\end{aligned}
$$

The constant $C_0 = C_0(n)$ does not depend on i. If $\varepsilon_0 = \varepsilon_0(n) > 0$ is chosen sufficiently small, we come up with

$$
u \ge \frac{1}{C} \mathcal{K}_n\varphi_0 > 0. \tag{4.6}
$$

The general case $\varphi_0 \in C^1(\partial B)$ follows from (4.6) with help of an approximation argument, the maximum estimates of [19] and local L^p-estimates [18]. ∎

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