Weakly Coupled Elliptic Systems and Positivity

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Abstract

In this paper we will study under which conditions the positive cone, or part of the positive cone, is preserved when solving a weakly coupled system of elliptic partial differential equations. Such a system will be as follows:

$$\begin{pmatrix} -\Delta_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & -\Delta_k \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_k \end{pmatrix} + \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$$

on a bounded domain in $\mathbb{R}^n$, with zero Dirichlet boundary condition. The operators Δ_i will be strictly elliptic such as the Laplacian. The system is said to preserve the positive cone if $f \geq 0$ implies $u \geq 0$. We will classify such systems. For noncooperative systems we need and show pointwise estimates for iterates of the Green function.

Contents

1 Introduction

A system of elliptic partial differential equations is called weakly coupled if there appear no derivatives in the coupling terms. In such a system a matrix equation and elliptic differential equations are combined:

the matrix equation:	$M\vec{v} = \vec{f}$	$\vec{f} \in I\!R^k$,	find $\vec{v} \in I\!R^k$;
the elliptic p.d.e.:	$Lw\,(x) = f\,(x)$	$f \in C(\bar{\Omega})$,	find $w \in C_0(\bar{\Omega})$;
the system:	$L\vec{u}\,(x) + M\vec{u}\,(x) = \vec{f}\,(x)$	$\vec{f} \in C(\bar{\Omega})^k$,	find $\vec{u} \in C_0(\bar{\Omega})^k$.

In the simplest combination, the so called cooperative systems, positivity properties of both matrix equation and elliptic differential equations support each other. Positivity follows by combining the results of both type of equations. In noncooperative systems the matrix involved does not preserve the positive cone. It might preserve a subcone of the positive cone or no cone at all. Roughly said, these cases correspond to the similar to cooperative and strictly noncooperative case. To have a positivity result in the last case the classical maximum principle for elliptic equations is not sufficient. In general the positive cone is not preserved. However, using pointwise estimates for Green functions, one might still have a positive solution if the source term lies in a subcone of the positive cone, or one might obtain positivity for some components of the solution for general positive source term. We will give such results.

Positivity results for cooperative systems can be found in the book by Protter and Weinberger [23], and in the papers of Walter [29], De Figueiredo and Mitidieri [12] as well as in [26], [19]. Problems that are similar to cooperative are studied by Weinberger in [30] and by Cosner and Schaefer in [7]. Results for noncooperative systems can be found in [25], [5], [6], [26]. A particular noncooperative system was studied in [11]. Positivity results for some systems that have coupling in the first order derivatives can be found in [27].

The following we will use throughout the paper.

M(ain) A(ssumptions):

1. domain: $\Omega \subset \mathbb{R}^n$, bounded and $\partial\Omega \subset C^{1,1}$,

2. elliptic operator: $L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c$,

with $c \geq 0$ and $\exists \alpha, \beta, \gamma > 0$ such that

$$\alpha\,|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \beta\,|\xi|^2,$$

$$a_{ij}, b_j, c \in C^{0,\gamma}(\bar{\Omega}),$$

$$\Delta = \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2},$$

3. matrix: $M = \begin{pmatrix} M_{11}(\cdot) & \cdots & M_{k1}(\cdot) \\ \vdots & & \vdots \\ M_{1k}(\cdot) & \cdots & M_{kk}(\cdot) \end{pmatrix}$, with $M_{i,j} \in C(\bar{\Omega})$,

4. ordering in $C(\bar{\Omega})^k$: $u \geq 0$ meaning $u_i(x) \geq 0 \quad \forall x \in \Omega, j = 1, \dots, k$,

$u > 0$ meaning $u_i(x) > 0 \quad \forall x \in \Omega, j = 1, \dots, k$,

5. ordering in $\mathcal{L}(C(\bar{\Omega})^k)$: $\mathcal{A} \geq 0$ meaning $u \geq 0 \Rightarrow \mathcal{A}u \geq 0 \;\; \forall u \in C(\bar{\Omega})^k$,

$\mathcal{A} > 0$ meaning $0 \neq u \geq 0 \Rightarrow \mathcal{A}u > 0 \; \forall u \in C(\bar{\Omega})^k$,

6. first eigenvalue/eigenfunction: $(\lambda_0, \phi_0) \in \mathbb{R} \times C(\bar{\Omega}; \mathbb{R})$, with

$$\begin{cases} L\phi_0 = \lambda_0\phi_0 & \text{in } \Omega, \\ \phi_0 = 0 & \text{on } \partial\Omega, \\ \phi_0 > 0. \end{cases}$$

L is a second order strictly elliptic differential operator. For its properties see [13].

We will end the introduction by recalling and comparing results for the matrix equation and the differential equation. In section 2 we will classify weakly coupled elliptic systems. Section 3 contains and explains necessary and sufficient conditions for cone preserving properties of a simple non trivial system. In section 4 we will obtain estimates for iterates of the Green function which we will need in the noncooperative case. In section 5 one finds results for the cooperative case. Such results have their own interest but we also need them in order to handle the noncooperative case. The noncooperative case finally is treated in section 6. We end by stating our result for the problem that was proposed by McKenna and Walter in [21]:

$$\left[\begin{array}{ll} \Delta^2 u + b\, u = f & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1}$$

There exists $c(\Omega) > 0$ such that, whenever $0 \leq b \leq c(\Omega)$, we have $f \geq 0$ implies $u \geq 0$. Note that (1) can be written as a system, namely, with $v = -\Delta u$:

$$\left[\begin{array}{ll} -\Delta u = v & \text{in } \Omega, \\ -\Delta v = -b\, u + f & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{2}$$

Also note that the classical maximum principle shows that for every $f \in C(\bar{\Omega})$ with $f > 0$ there exists a $c_f > 0$ such that for $0 \leq b \leq c_f$ one finds $u > 0$. This maximum principle however does not show that $\inf \{c_f; f \in C(\bar{\Omega})^+\} > 0$.

1.1 Two model operators

Let us consider

i) L an elliptic operator as in **MA 2.,**

and

ii) M a constant real $k \times k$-matrix, with $M = sI - B$,

such that 1) $B_{ij} \geq 0$ for all i, j.
2) $s > \rho(B)$, the spectral radius,
3) B irreducible.

The matrix M as in ii) that satisfies ii.1) and $s \geq \rho(B)$ is called an M-matrix. An M-matrix M is nonsingular if and only if $s > \rho(B)$. B is irreducible if and only if M is irreducible. See [3].

A finite element approximation of the operator L with appropriate boundary conditions gives such an M-matrix.

The next subsection contains some common properties of these two classes of operators.

1.2 Classical results

Let Op either stand for

$$L : \mathcal{D}(L) = C_0\left(\overline{\Omega}; I\!R\right) \cap C^{2+\varepsilon}\left(\overline{\Omega}; I\!R\right) \to C^{\varepsilon}\left(\overline{\Omega}; I\!R\right), \tag{3}$$

with L as in **i)**, or for

$$M : \mathcal{D}(M) = I\!R^n \to I\!R^n \tag{4}$$

with M as in **ii)**.

We will consider the model equation

$$Op\ u = \lambda u + f. \tag{5}$$

For all $f \in Im\,(Op)$ one can find a unique $u \in \mathcal{D}\,(Op)$ that solves (5) if and only if $\lambda \notin \sigma\,(Op)$, where $\sigma\,(Op) \in \mathbb{C}$ denotes the spectrum. Since the inverse of Op is compact, $\sigma\,(Op)$ is discrete and consists of at most countably many eigenvalues.

A common feature concerning the spectrum of both operators is the following.

A. Set $\lambda_0 = \inf\{Re\lambda; \lambda \in \sigma\,(Op)\}$. Then λ_0 is an eigenvalue, $\lambda_0 > 0$ and the corresponding eigenfunction/eigenvector u_0 is unique up to normalization. Moreover, $u_0 > 0$ and it is the only positive eigenfunction/eigenvector.

And with respect to positivity they have the following (related) properties in common.

B. There is $\lambda_0 > 0$ such that

1. **a.** For all $\lambda \in (-\infty, \lambda_0)$ one finds $(Op - \lambda)^{-1}$ is positive:

   $$f \geq 0 \quad \text{implies} \quad u \geq 0.$$

   b. and even that $(Op - \lambda)^{-1}$ is strongly positive:

   $$0 \neq f \geq 0 \quad \text{implies} \quad u > 0.$$

2. If $\lambda = \lambda_0$,

   $$0 \neq f \geq 0 \quad \text{implies nonexistence for } u.$$

3. For all $\lambda \in (\lambda_0, \infty)$ one finds that if there exists a solution u, then

   $$0 \neq f \geq 0 \quad \text{implies} \quad u \not\geq 0.$$

The assertion in (**A**) follows from the Krein-Rutman Theorem. The older finite dimensional result, which is sufficient for the matrix case, is due to the Perron-Frobenius Theorem. Both theorems use positivity of the inverse operator Op^{-1}. Krein-Rutman uses the compact imbedding $\mathcal{D}\,(Op)$ in $Im\,(Op)$.

For elliptic operators assertion (**B.1.a**) is the maximum principle and (**B.1.b**) the strong maximum principle.

2 Classification of weakly coupled elliptic systems

2.1 Cooperative elliptic systems

A combination of the two operators appears in the following weakly coupled elliptic system:

$$(I_k \times L)\, u + Mu = \lambda u + f \tag{6}$$

where u and f are vector functions, and

$$I_k \times L = \begin{pmatrix} L & 0 & & 0 \\ 0 & L & & 0 \\ & & & \\ 0 & 0 & & L \end{pmatrix} : \left(C_0(\bar{\Omega}) \cap C^{2+\varepsilon}(\bar{\Omega})\right)^k \to \left(C^{\varepsilon}(\bar{\Omega})\right)^k \tag{7}$$

If M is a irreducible M-matrix and L the elliptic operator above then it has been shown that *Op* defined by

$$I_k \times L + M : \left(C_0(\bar{\Omega}) \cap C^{2+\varepsilon}(\bar{\Omega})\right)^k \to \left(C^{\varepsilon}(\bar{\Omega})\right)^k \tag{8}$$

has the same properties (**A**) and (**B**). See [28], [12], [26] and [19].

2.1.1 Generalizations

One does not need the same elliptic operator on every diagonal term. The properties (**A**) and (**B**) also hold for systems where $I_n \times L$ is replaced by

$$\mathrm{L} = \begin{pmatrix} L_1 & 0 & & 0 \\ 0 & L_2 & & 0 \\ & & & \\ 0 & 0 & & L_k \end{pmatrix}, \tag{9}$$

where the L_m, with $m \in \{1, \ldots, k\}$, are general second order strictly elliptic operators with sufficiently regular (real) coefficients such that $\sigma\left(L_m\right) \subset I\!R^+ \times iI\!R$.

One may also replace the constant matrix M by a matrix $M = M\left(x\right)$ such that $M\left(x\right)$ is an irreducible M-matrix for every $x \in \Omega$. The irreducibility may be weakened to fully coupled ([26]).

Definition 2.1 *We call the matrix M* **fully coupled** *if the following holds. For all $\alpha, \beta \subset \{1, \ldots, k\}$ with $\alpha \cup \beta = \{1, \ldots, k\}$ and $\alpha \cap \beta = \emptyset$, there is $i \in \alpha, j \in \beta$ and $x \in \Omega$ such that $M_{ij}\left(x\right) \neq 0$.*

Definition 2.2 *An elliptic system is called* **cooperative** *if the differential equation has the form*

$$\left(\mathrm{L} + M - \lambda I_k\right) u = f. \tag{10}$$

with L *as in (9) and $M\left(x\right)$ being an M-matrix for every $x \in \Omega$.*

In summary: under sufficient regularity of L and M the properties (**A**) and (**B**) hold for *Op* defined by

$$\mathrm{L} + M : \left(C_0(\bar{\Omega}) \cap C^{2+\varepsilon}(\bar{\Omega})\right)^k \to \left(C^\varepsilon(\bar{\Omega})\right)^k \tag{11}$$

with L and M as above.

2.2 Elliptic systems similar to cooperative

Since diagonal operators $I_k \times L$ as in (7) commute with constant square matrices S one may look for cone preserving properties of $(I_k \times L + M)^{-1}$ in other cones as the positive one. Suppose that S is invertible. Since $I_k \times L = S^{-1} S\,(I_k \times L) = S^{-1}\,(I_k \times L)\,S$ the system in (6) is then replaced by

$$(I_k \times L)\,Su + \left(SMS^{-1}\right) Su = \lambda Su + Sf. \tag{12}$$

If SMS^{-1} is a irreducible M-matrix, then for $\lambda < \lambda_1$ one finds that

$$\begin{cases} Sf \geq 0 & \text{implies} \quad Su \geq 0, \\ 0 \neq Sf \geq 0 & \text{implies} \quad Su > 0. \end{cases}$$

Here λ_1 denotes the first eigenvalue of *Op* defined by

$$I_k \times L + SMS^{-1} : \left(C_0\left(\overline{\Omega}\right) \cap C^{2+\varepsilon}\left(\overline{\Omega}\right)\right)^k \to \left(C^\varepsilon\left(\overline{\Omega}\right)\right)^k. \tag{13}$$

If there is a matrix S such that SMS^{-1} has only nonpositive off-diagonal entries, then system (6) is called similar to cooperative.

A disadvantage of this approach is that the preserved cones of system (12) may have little in common with the positive cone. Conditions such that a constant matrix M can be transformed to a matrix SMS^{-1} have been studied in [30].

2.2.1 Generalizations

Again one may replace the constant matrix M with $M(x)$. However, since L as in (9) in general does not commute with matrices S one cannot replace $I \times L$ with L straightforwardly.

Instead of using square matrices S one may even use real $k' \times k$-matrices S, with $k' > k$, that have a left inverse T. The system in (6) is then replaced by

$$(I_{k'} \times L)\,Su + (SMT)\,Su = \lambda Su + Sf. \tag{14}$$

Such an approach is found in [20].

2.3 Noncooperative elliptic systems

Systems as (6) where the coupling matrix M has at least one positive off-diagonal entry are called noncooperative. In general such a non cooperative system is not similar to a cooperative system. For example, systems of two equations are similar to cooperative if and only if the coupling matrix has (two) real eigenvalues.

For noncooperative systems one cannot expect to find the results that are stated in the properties (**A**) and (**B**). See Proposition 3.6 in the next section. Nevertheless, using pointwise estimates for the Green function belonging to the elliptic operator, one can still find a restricted positivity result. This will be the subject of a main part of the paper. In the most general case M is replaced with $M(x) = M^+(x) - M^-(x)$ with $M^+_{i,j}(x), M^-_{ij}(x) \geq 0$ (the system is cooperative if and only if $M^+_{ij}(x) = 0$ for all $i \neq j$).

3 The simplest nontrivial example

In this section we consider a two-equations system with constant coefficients:

$$\left[\begin{array}{ll} -\Delta u_1 = au_1 + bu_2 + f_1 & \text{in } \Omega, \\ -\Delta u_2 = cu_1 + du_2 + f_2 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{15}$$

We will give algebraic conditions that are necessary and sufficient for existence of an invariant cone, existence of a positive eigenfunction, and for having a maximum principle. We will concentrate on the results. For sake of completeness the proofs can be found at the end of this section. Results for such a system can also be found in [30].

3.1 Results

We assume that the system in (15) is irreducible: $bc \neq 0$, and we will restrict ourselves to the case that one coupling coefficient is positive: $c > 0$. Since the elliptic operators in both components are the same (and since the coefficients are constant) we have a real decoupling of (15) when

$$\tfrac{1}{4}(a-d)^2 + bc > 0. \tag{16}$$

Now elementary linear algebra yields that (15) is equivalent with

$$\left[\begin{array}{ll} -\Delta \times Su = \begin{pmatrix} \frac{1}{2}(a+d) + w & 0 \\ 0 & \frac{1}{2}(a+d) - w \end{pmatrix} Su + Sf & \text{in } \Omega, \\ Su = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{17}$$

where

$$S = \frac{1}{2cw} \begin{pmatrix} c & w - \frac{1}{2}(a-d) \\ -c & w + \frac{1}{2}(a-d) \end{pmatrix} \text{ and } w = \sqrt{\tfrac{1}{4}(a-d)^2 + bc}.$$

The eigenvectors of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are

$$\upsilon^{+} = \begin{pmatrix} \frac{1}{2}(a-d)+w \\ c \end{pmatrix} \text{ and } \upsilon^{-} = \begin{pmatrix} \frac{1}{2}(a-d)-w \\ c \end{pmatrix}. \tag{18}$$

They respectively correspond with the eigenvalues $\mu^{+} = \frac{1}{2}(a+d)+w$ and $\mu^{-} = \frac{1}{2}(a+d)-w$. The spectrum for (15) consists of eigenvalues: $\sigma_d\left(-\Delta \times I - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = \sigma_d\left(-\Delta - \mu^{+}\right) \cup \sigma_d\left(-\Delta - \mu^{-}\right)$. Here d stands for Dirichlet boundary condition.

Definition 3.1 *For $\alpha, \beta \in I\!R^2 \setminus \{0\}$ we set*

$$K(\alpha,\beta) = \left\{ f \in C(\bar{\Omega})^2;\ \alpha \cdot f \geq 0,\ \beta \cdot f \geq 0 \right\}.$$

Definition 3.2 *We call $K(\alpha,\beta)$ an invariant cone for (15) if for all $f \in K(\alpha,\beta)$ the system has a unique solution u and $u \in K(\alpha,\beta)$.*

Now let (λ_0, ϕ_0) denote the first eigenvalue of $-\Delta\phi = \lambda\,\phi$ on Ω with $u = 0$ on $\partial\Omega$.

Proposition 3.3 (Existence of invariant cone) *Let $c > 0$. The following are equivalent.*

1. $\frac{1}{4}(a-d)^2 + bc \geq 0$ *and* $\lambda_0 > \frac{1}{2}(a+d) + \sqrt{\frac{1}{4}(a-d)^2 + bc}$,
2. *There exist independent $\alpha, \beta \in I\!R^2$ such that $K(\alpha,\beta)$ is an invariant cone for (15).*

It will be useful to have an algebraic condition in order to see which cones are invariant.

Lemma 3.4 *Let $c > 0$ and suppose that*

$$\tfrac{1}{4}(a-d)^2 + bc > 0 \quad \text{and} \quad \lambda_0 > \tfrac{1}{2}(a+d) + \sqrt{\tfrac{1}{4}(a-d)^2 + bc}.$$

Let (μ^{+}, υ^{+}) and (μ^{-}, υ^{-}) be as in (18) and fix independent vectors $\alpha, \beta \in I\!R^2$. Then the following two statements are equivalent.

1. (a) $\upsilon^{+} \in K(\alpha,\beta)$ *or* $-\upsilon^{+} \in K(\alpha,\beta)$, *and*
   (b) $\upsilon^{-} \notin K(\alpha,\beta)^{\circ}$ *and* $-\upsilon^{-} \notin K(\alpha,\beta)^{\circ}$.
2. $K(\alpha,\beta)$ *is an invariant cone for (15).*

Remark 1: If we replace $\frac{1}{4}(a-d)^2+bc>0$ by $\frac{1}{4}(a-d)^2+bc=0$ we still have that *2.* implies *1.* In this case $v^+=v^-$ and *1.* means $v^+\in\partial K(\alpha,\beta)$ or $-v^+\in\partial K(\alpha,\beta)$.

Next we consider the eigenvalue problem

$$\left[\begin{array}{ll} -\Delta\Phi=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\Phi+\lambda\,\Phi & \text{in }\Omega,\\ \Phi=0 & \text{on }\partial\Omega.\end{array}\right. \tag{19}$$

Proposition 3.5 (Existence of a positive eigenfunction) *Let $c>0$. The following are equivalent.*

1. $\frac{1}{4}(a-d)^2+bc\geq 0$ *and* $\frac{1}{2}(a-d)+\sqrt{\frac{1}{4}(a-d)^2+bc}\geq 0.$
2. *There exist a positive eigenfunction of (19).*

Remark 2: The claim from *1.* to *2.* follows immediately by an explicit formula. With (λ_0,ϕ_0) as above a positive eigenfunction Φ_0 with its eigenvalue Λ_0 is defined by

$$\left\{\begin{array}{l} \Lambda_0=\lambda_0-\frac{1}{2}(a+d)-\sqrt{\frac{1}{4}(a-d)^2+bc},\\ \Phi_0=\begin{pmatrix}\frac{1}{2}(a-d)+\sqrt{\frac{1}{4}(a-d)^2+bc}\\ c\end{pmatrix}\phi_0.\end{array}\right. \tag{20}$$

Proposition 3.6 (Maximum principle) *Let $c>0$. The following are equivalent.*

1. $b\geq 0$ *and* $\lambda_0>\frac{1}{2}(a+d)+\sqrt{\frac{1}{4}(a-d)^2+bc},$
2. $K\left(\binom{1}{0},\binom{0}{1}\right)$ *is an invariant cone for (15).*

Remark 3: Note that in order to have this maximum principle one needs the algebraic conditions of Proposition 3.3, Proposition 3.5 and $b>0$. (The inequality $\frac{1}{2}(a-d)+\sqrt{\frac{1}{4}(a-d)^2+bc}\geq 0$ is implied by $bc\geq 0$.)

Remark 4: A positive eigenfunction of (19) is a supersolution for (15) with $f=0$. Hence, if $\frac{-(a-d)^2}{4c}<b<0$ there is a positive supersolution but the system does not preserve the positive cone. This fact denies the conjecture that existence of a positive supersolution implies the maximum principle.

Remark 5: Suppose that $c>0$. Then $b>0$ makes the system cooperative and $\frac{1}{4}(a-d)^2+bc>0$ makes the system similar to cooperative. If $b<0$ or even $\frac{1}{4}(a-d)^2+bc<0$ holds, the best positivity results that one might hope for are as follows. i) For some

$K^* \subset K\left(\binom{1}{0},\binom{0}{1}\right)$ one has: $f \in K^*$ implies $u \in K\left(\binom{1}{0},\binom{0}{1}\right)$. ii) For some $K^{**} \supset K\left(\binom{1}{0},\binom{0}{1}\right)$ one has $f \in K\left(\binom{1}{0},\binom{0}{1}\right)$ implies $u \in K^{**}$. If $b < 0$ but $\frac{1}{4}(a-d)^2 + bc > 0$ holds such a result follows from Proposition 3.3 (the similar to cooperative case). If $\frac{1}{4}(a-d)^2 + bc < 0$ (the strictly noncooperative case) one needs a different method. Such a type of result is found in Theorem 6.4.

3.2 Proofs

Proof of Lemma 3.4. ($1 \Rightarrow 2$) Suppose that Proposition 3.3 1 holds. For $f \in C(\bar{\Omega})^2$, there are unique $g_1, g_2 \in C(\bar{\Omega})$ with $f = \upsilon^+ g_1 + \upsilon^- g_2$. Because of assumption 1.a) and 1.b) we find $g_1 \geq 0$. Since $\lambda_0 > \mu^+ > \mu^-$ we obtain that $(-\Delta - \mu^+)^{-1}$ and $(-\Delta - \mu^-)^{-1} : C(\bar{\Omega}) \to C_0(\bar{\Omega})$ are well-defined positive operators. We even have $(-\Delta - \mu^+)^{-1} \geq (-\Delta - \mu^-)^{-1} \geq 0$. Hence

$$u = \upsilon^+ (-\Delta - \mu^+)^{-1} g_1 + \upsilon^- (-\Delta - \mu^-)^{-1} g_2 =$$
$$= \upsilon^+ \left((-\Delta - \mu^+)^{-1} - (-\Delta - \mu^-)^{-1} \right) g_1 + (-\Delta - \mu^-)^{-1} (\upsilon^+ g_1 + \upsilon^- g_2).$$

Since $g_1 \geq 0$ and $\upsilon^+ \in K(\alpha,\beta)$ we obtain $\upsilon^+ \left((-\Delta - \mu^+)^{-1} - (-\Delta - \mu^-)^{-1} \right) g_1 \in K(\alpha,\beta)$. Since $\upsilon^+ g_1 + \upsilon^- g_2 \in K(\alpha,\beta)$ we obtain $(-\Delta - \mu^-)^{-1} (\upsilon^+ g_1 + \upsilon^- g_2) \in K(\alpha,\beta)$, and hence $u \in K(\alpha,\beta)$.

($2 \Rightarrow 1$) Now suppose 1.a or 1.b is not satisfied. Then there are c_1, c_2 such that $c_1\upsilon^+ + c_2\upsilon^- \in \partial K(\alpha,\beta) \setminus \{0\}$ and $c_1\upsilon^+ + \vartheta c_2 \upsilon^- \notin K(\alpha,\beta)$ for all $\vartheta \in [0,1)$. Solving for $f = (c_1\upsilon^+ + c_2\upsilon^-)\phi_0$ we obtain

$$u = (\lambda_0 - \mu^+)^{-1} \left(c_1\upsilon^+ + \left(1 - \frac{\mu^+ - \mu^-}{\lambda_0 - \mu^-}\right) c_2 \upsilon^- \right) \phi_0 \notin K(\alpha,\beta). \qquad \square$$

Proof of Proposition 3.3. ($1 \Rightarrow 2$) First suppose that $\frac{1}{4}(a-d)^2 + bc > 0$. Hence there exist independent υ^+, υ^- as in (18). By Lemma 3.4 we find that the cone $K(\upsilon^+ + \upsilon^-, \upsilon^+ - \upsilon^-)$ is invariant. If $\frac{1}{4}(a-d)^2 + bc = 0$ we have that $\upsilon^+ = \upsilon^-$. Let $\upsilon^\perp \in I\!R^2$ be perpendicular to υ^+. Then either $K\left(\upsilon^+, \upsilon^\perp\right)$ or $K\left(\upsilon^+, -\upsilon^\perp\right)$ is invariant.

($2 \Rightarrow 1$) Suppose that there is an invariant cone $K(\alpha,\beta)$ with α, β independent. Since $K(\alpha,\beta)$ contains an nonempty open set one finds that $0 \notin \sigma\left(-\Delta I - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)$. Write $\mathcal{R} = \left(-\Delta I - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)^{-1}$ and $\tilde{K} = K(\alpha,\beta) \cap I\!R^2$. For $\eta \in \tilde{K}$ we have $\mathcal{R}\eta\phi_0 = \phi_0 \left(\begin{smallmatrix} a-\lambda_0 & b \\ c & d-\lambda_0 \end{smallmatrix}\right)^{-1} \eta \in \phi_0 \tilde{K}$. Since $\tilde{K}$ is convex the fixed point Theorem shows that there exists an eigenvector in $\phi_0\tilde{K}$ of $-\Delta I - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with eigenvalue $\lambda > 0$. Since the eigenvalues\functions of $\left(\begin{smallmatrix} a-\lambda_0 & b \\ c & d-\lambda_0 \end{smallmatrix}\right)$ are $(\mu^+ - \lambda_0, \upsilon^+)$ and $(\mu^- - \lambda_0, \upsilon^-)$ we have $\eta = \upsilon^+$ and $\lambda = \lambda_0 - \mu^+ \in I\!R^+$. Condition 1 follows from $\lambda_0 - \mu^+ \in I\!R^+$. $\square$

Proof of Proposition 3.5 ($2 \Rightarrow 1$) Suppose there exists an eigenfunction and $\frac{1}{4}(a-d)^2 + bc < 0$ holds. Then from (17) we obtain a complex eigenvalue for the Laplacian with Dirichlet boundary condition, a contradiction. Now suppose that $\frac{1}{4}(a-d)^2 + bc \geq 0$. Since the Laplacian has just one eigenfunction with a fixed sign, the only eigenfunctions

Φ of $-\Delta I - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $\Phi(x) \in K(\alpha, \beta)$ for all $x \in \Omega$, are $\upsilon^+\phi_0$ and $\upsilon^-\phi_0$. Hence one of these two should be positive. Since $\upsilon^- \geq 0$ implies $\upsilon^+ \geq 0$ (see (18)) we find that $\frac{1}{2}(a-d) + \sqrt{\frac{1}{4}(a-d)^2 + bc} \geq 0$. □

Proof of Proposition 3.6 ($1 \Rightarrow 2$) If $b > 0$ one finds from (18) that $\upsilon^+ \geq 0$ and $\upsilon^- \not\geq 0$. By Lemma 3.4 the positive cone is invariant.

($2 \Rightarrow 1$) From Proposition 3.3 one finds $\frac{1}{4}(a-d)^2 + bc \geq 0$ and $\lambda_0 > \frac{1}{2}(a+d) + \sqrt{\frac{1}{4}(a-d)^2 + bc}$. If $\frac{1}{4}(a-d)^2 + bc > 0$ Lemma 3.4 shows that $\upsilon^+ \in K\left(\binom{1}{0}, \binom{0}{1}\right)$ and $\upsilon^- \notin K\left(\binom{1}{0}, \binom{0}{1}\right)^\circ$. Hence $\frac{1}{2}(a-d) + \sqrt{\frac{1}{4}(a-d)^2 + bc} \geq 0$ and $\frac{1}{2}(a-d) - \sqrt{\frac{1}{4}(a-d)^2 + bc} \leq 0$. This is equivalent with $\frac{1}{2}|a-d| \leq \sqrt{\frac{1}{4}(a-d)^2 + bc}$. Since $c > 0$ it implies that $b \geq 0$. If $\frac{1}{4}(a-d)^2 + bc = 0$ one has $\upsilon^+ = \upsilon^- \in \partial K\left(\binom{1}{0}, \binom{0}{1}\right)$ and hence $a = d$. Consequently we find $b = 0$. □

4 The Green function

4.1 Motivation

If the first eigenvalue/function (λ_0,ϕ_0) of

$$\left[\begin{array}{ll} L\phi = \lambda\ \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega \end{array}\right. \tag{21}$$

satisfies $\lambda_0 > 0$ (guaranteed by positivity of the last coefficient of L: $c(\cdot) \geq 0$), then the solution of

$$\left[\begin{array}{ll} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{array}\right. \tag{22}$$

can be written as

$$u(x) = \int_\Omega G(x, y) f(x)\ dx.$$

We will denote $u = \mathcal{G}f$. The operator $\mathcal{G}$ is strictly positive on $C(\bar{\Omega})$. By the strong maximum principle one finds that if $f \in C(\bar{\Omega})$ then $0 \leq f \neq 0$ implies $\mathcal{G}f(x) > 0$ for all $x \in \Omega$. Then $\mathcal{G}^k$ or any nontrivial polynomial in $\mathcal{G}$ with positive coefficients has the same property. But what can be said about positivity for general polynomials in $\mathcal{G}$? Is it allowed to have some small negative coefficients? It has been shown, see [1], [32], [33] and [9], that there is $c > 0$ such that $\mathcal{G}^2 \leq c\ \mathcal{G}$. It can also been shown that $\mathcal{G} \not\leq c\ \mathcal{G}^2$ for any c in any dimension.

In this section we will show that there is a $k_n \in I\!N$ such that $\mathcal{G}^{k_n}$ can be estimated from above and from below by a constant times the projection on the first eigenfunction. In other words, there are $c_1, c_2 > 0$ such that for all $f \in C(\bar{\Omega})^+, x \in \Omega$:

$$c_1\ \langle\phi_0, f\rangle\ \phi_0(x) \leq \left(\mathcal{G}^{k_n} f\right)(x) \leq c_2\ \langle\phi_0, f\rangle\ \phi_0(x),$$

where

$$\langle \phi_0, f \rangle = \int_\Omega f(y) \ \phi_0(y) \ dy.$$

As a consequence we have for all $k \geq k_n$ that

$$\frac{c_1}{c_2} \mathcal{G}^{k_n} \leq \lambda_0^{k_n - k} \mathcal{G}^k \leq \frac{c_2}{c_1} \mathcal{G}^{k_n}, \tag{23}$$

and hence that some small negative coefficients in a polynomial in $\mathcal{G}$ can be compensated for. That means, the polynomial operator is still positive. In fact (23) allows us to show positivity for power series in $\mathcal{G}$ when the negative coefficients are small compared with the positive ones. Without this result one is not able to handle systems when its components contain different elliptic operators. In [26] the positivity result for noncooperative systems was shown only for systems with the same elliptic operator in every component.

4.2 Estimates for the iterated Green function

Let L be an elliptic operator satisfying **MA 2.** The case $n = 2$ is somewhat special and in order to use a result from [1] we need additional regularity, namely $a_{ij}, b_i, c \in C^{1,\alpha}(\bar{\Omega})$. Again let $G(\cdot, \cdot)$ denote the Green function. Define

$$\begin{aligned} G^1(x,y) &= G(x,y) \\ G^{k+1}(x,y) &= \int_{z \in \Omega} G(x,z) \ G^k(z,y) \, dz \quad \text{for } k = 1, 2, \ldots . \end{aligned}$$

Then by changing the order of integration we find for $k = 1, 2, \ldots$ and $f \in C(\bar{\Omega})$ that

$$\left(\mathcal{G}^k f\right)(x) = \int_{y \in \Omega} G^k(x,y) f(y) \ dy.$$

Theorem 4.1 (3G-Theorem of Cranston-Fabes-Zhao) *There is a constant $\lambda_c > 0$ such that*

$$G(x,y) \geq \lambda_c \ G^2(x,y) \qquad \textit{for all } x, y \in \Omega. \tag{24}$$

Remark 1: The general statement for $n \geq 3$ is found in [9]. For the Laplacian see [32]. For the case $n = 2$ see [1] and [33]. See also [17] and [27].

Theorem 4.2 *Let L and Ω satisfy the assumptions in* **MA**. *Then there are $c_1, c_2 > 0$ such that, with $k_n = \left[\frac{n+2}{2}\right] + 1$, the k_n^{th}-iterated Green function satisfies for all $x, y \in \Omega$*

$$c_1 \ \phi_0(x) \ \phi_0(y) \leq G^{k_n}(x,y) \leq c_2 \ \phi_0(x) \ \phi_0(y). \tag{25}$$

Remark 2: We will not prove it but the theorem also holds for $n = 1$. For $\Omega = (-1, 1)$ and $Lu = u''$ one finds

$$\begin{aligned} &G(x,y) = \tfrac{1}{2}\left(1 - |x-y| - xy\right), \\ &G^2(x,y) = \tfrac{1}{12}\left(2 + 2|x-y| - x^2 - y^2\right)\left(1 - |x-y| - xy\right), \\ &\phi_0(x) = \cos\left(\tfrac{\pi}{2}\, x\right). \end{aligned}$$

Elementary calculus shows that (25) holds with $k_1 = 2$ and also that (25) does not hold when we replace k_1 by 1. Hence the number k_1 is optimal. We expect the number k_n to be optimal in every dimension.

Remark 3: Both theorems combined show that the following ordering exists:

$$\mathcal{G} \succ \mathcal{G}^2 \succ \mathcal{G}^3 \succ \ \ldots \ \succ \mathcal{G}^{k_n-1} \succ \mathcal{G}^{k_n} \sim \mathcal{G}^{k_n+1} \sim \mathcal{G}^{k_n+2} \sim \ \ldots \quad ,$$

with $\mathcal{A} \succ \mathcal{B}$ meaning: there is $c > 0$ such that $c\mathcal{A} > \mathcal{B}$, and $\mathcal{A} \sim \mathcal{B}$ meaning: there is $c > 0$ such that $c\mathcal{A} > \mathcal{B} > c^{-1}\mathcal{A}$.

Proof. The estimate from above can be found by using regularity theory or by doing some singular integral calculus and using pointwise estimates for the Green function. We will use the first approach. Redoing the singular integral calculus is more tedious but will show the best k_n.

By results of Widman [31], Zhao [32] and Hueber-Sieveking [16], [17] the Green function for $n \geq 3$ satisfies for some $c_{\Omega,1}, c_{\Omega,2} > 0$

$$c_{\Omega,1}\ F_n(x,y) \leq G(x,y) \leq c_{\Omega,2}\ F_n(x,y) \qquad \text{for all } x, y \in \Omega, \tag{26}$$

with

$$F_n(x,y) = |x-y|^{2-n} \min\left(1, \frac{d(x)\ d(y)}{|x-y|^2}\right) \tag{27}$$

and $d(x) \equiv d(x, \partial\Omega)$. By [31], [1] and [33] the Green function for $n = 2$ satisfies (26) with

$$F_2(x,y) = \log\left(1 + \frac{d(x)\ d(y)}{|x-y|^2}\right). \tag{28}$$

Let $x^*, y^* \in \partial\Omega$ be such that $|x - x^*| = d(x), |y - y^*| = d(y)$. Then $d(x) = |x - x^*| \leq |x - y^*| \leq |x - y| + |y - y^*| = |x - y| + d(y)$. Using that $\min(1, a^2 + a) \leq 2a$ for $a \geq 0$, we find

$$\min\left(1, \tfrac{d(x)\ d(y)}{|x-y|^2}\right) \leq \min\left(1, \tfrac{d(y)}{|x-y|} + \tfrac{d(y)^2}{|x-y|^2}\right) \leq 2\tfrac{d(y)}{|x-y|} \tag{29}$$

and similarly for $n = 2$

$$\log\left(1 + \tfrac{d(x)\ d(y)}{|x-y|^2}\right) \leq \log\left(1 + \tfrac{d(y)}{|x-y|}\right)^2 \leq 2\tfrac{d(y)}{|x-y|}. \tag{30}$$

Hence for all $n \geq 2$ there is $c > 0$ such that

$$G(x,y) \leq c\ |x-y|^{1-n}\, d(y).$$

Then

$$\sup_{y\in\Omega}\left\|d\left(y\right)^{-1}G\left(\cdot,y\right)\right\|_{L^{p}(\Omega)}<\infty\qquad\text{for } p<\frac{n}{n-1}.$$

Using the regularity results for elliptic differential equations, see Theorem 9.19 of [13], we find for $k\geq 1$ that

$$\sup_{y\in\Omega}\left\|d\left(y\right)^{-1}G^{k}\left(\cdot,y\right)\right\|_{W^{2k-2,p}(\Omega)}<\infty\qquad\text{for } p<\frac{n}{n-1}.$$

By the Sobolev imbedding Theorem, see page 158 of [13], we obtain if $2k-2-\dfrac{n}{p}>1$ that

$$\sup_{y\in\Omega}\left\|d\left(y\right)^{-1}G^{k}\left(\cdot,y\right)\right\|_{C^{1}(\bar{\Omega})}<\infty. \tag{31}$$

Hence (31) holds when $k>\frac{1}{2}\left(n+2\right)$. Since $d\left(y\right)^{-1}G^{k}\left(x,y\right)=0$ for $y\in\Omega, x\in\partial\Omega$ and since $\partial\Omega\in C^{1,\alpha}$, we obtain that there exists $c>0$ such that

$$d\left(y\right)^{-1}G^{k}\left(x,y\right)\leq c\ d\left(x\right)\qquad\text{for all } x,y\in\Omega. \tag{32}$$

The estimate from above follows since for $C^{1,\alpha}$-domains there exist $c_1,c_2>0$ with

$$c_1\ d\left(x\right)\leq\phi_0\left(x\right)\leq c_2\ d\left(x\right)\qquad\text{for all } x\in\Omega. \tag{33}$$

For the estimate from below fix compact $K\subset\subset\Omega$ with $K^{\circ}\neq\emptyset$. Then we find that $\inf\limits_{x\in K}d\left(x\right)>0$ and that there exists $c>0$ such that, both for $n\geq 3$ and $n=2$, we have

$$\begin{aligned} G\left(x,y\right)&\geq c\ d\left(x\right)\qquad\text{for all } x\in\Omega,y\in K,\\ G\left(x,y\right)&\geq c\ d\left(y\right)\qquad\text{for all } x\in K,y\in\Omega. \end{aligned}$$

A straightforward estimate gives $c^{*},c^{**}>0$ such that

$$G^{k}\left(x,y\right)\geq c^{*}\int\limits_{K}\cdots\int\limits_{K}d\left(x\right)\ 1\ \cdots\ 1\ d\left(y\right)dz_1\ldots dz_{k-1}=c^{**}\ d\left(x\right)\ d\left(y\right)$$

for all $x,y\in\Omega$. The proof finishes by (33). □

Instead of (22) it will be useful to consider

$$\left[\begin{array}{ll} Lu=a\left(\cdot\right)\ f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{array}\right. \tag{34}$$

and the corresponding eigenvalueproblem

$$\left[\begin{array}{ll} Lu=\lambda\ a\left(\cdot\right)\ u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega. \end{array}\right. \tag{35}$$

The Green function for (34) is $G_a\left(x,y\right)=G\left(x,y\right)a\left(y\right)$. Let $\left(\lambda_a,\phi_a\right)$ denote the first (positive) eigenvalue\function of 35.

Lemma 4.3 *Let $a \in C(\bar{\Omega})$ such that $0 \le a \ne 0$. Then there are $c_1, c_2 > 0$ such that*

$$c_1 \ (\mathcal{G}_a)^{k_n - 1} \ \mathcal{G} \le \mathcal{G}^{k_n} \le c_2 \ (\mathcal{G}_a)^{k_n - 1} \ \mathcal{G}.$$

Proof. The first estimate follows for $c_1 = \|a\|_\infty^{1-k_n}$. For the second estimate take a compact $K \subset \Omega$ such that $a(x) \ge \alpha > 0$ for $x \in K$. For $n \ge 3$ one finds by using (32) that there exist $c, c' > 0$ such that

$$G^{k_n}(x,y) \le c \ d(x) \, d(y) \le$$

$$\le c' \int \cdots \int_{z_1,\dots,z_{k_n-1} \in K} |x - z_1|^{2-n} \min\left(1, \tfrac{d(x)}{|x-z|}\right) \ a(z_1) \ \left(\prod_{k=2}^{k_n-1} |z_k - z_{k-1}|^{2-n} \ a(z_k) \right) \dots$$

$$\dots \min\left(1, \tfrac{d(y)}{|z_{k_n-1}-y|}\right) |z_{k_n-1} - y|^{2-n} \ dz_1 \dots dz_{k_n-1}$$

and replacing K with Ω we may continue by

$$\le c'' \int \cdots \int_{z_1,\dots,z_{k_n-1} \in \Omega} G(x, z_1) \ a(z_1) \ \left(\prod_{k=2}^{k_n-1} G(z_{k-1}, z_k) \ a(z_k) \right) G(z_{k_n-1}, y) \, dz_1 \dots dz_{k_n-1} =$$

$$= c'' \int_{z \in \Omega} G_a^{k_n-1}(x,z) \ G(z,y) \ dz.$$

Similar arguments show the lemma for $n = 2$. □

Remark 4: It follows that for all $x, y \in \Omega$

$$c_1 \ \phi_0(x) \ \phi_0(y) \ a(y) \le G_a^{k_n}(x,y) \le c_2 \ \phi_0(x) \ \phi_0(y) \ a(y). \tag{36}$$

Since there are $c_1, c_2 > 0$ such that $c_1 \phi_0 \le \phi_a < c_2 \phi_0$ we may replace ϕ_0 by ϕ_a in (36). It is also allowed to have different L and a at successive steps. Denoting $G_i(x,y) = G_{L_i}(x,y) \ a_i(y)$, $i = 1, \dots, k_n$ and $\widetilde{\mathcal{G}}^k = \mathcal{G}_k \mathcal{G}_{k-1} \dots \mathcal{G}_1$ one may prove that there exist c_1,$c_2 > 0$ such that for all $x, y \in \Omega$ we have

$$c_1 \ \phi_0(x) \ \phi_0(y) \ a_1(y) \le G^{k_n}(x,y) \le c_2 \ \phi_0(x) \ \phi_0(y) \ a_1(y). \tag{37}$$

5 Results for cooperative systems

5.1 Eigenvalue problems

Let L, as a (9), and M be such that the system is fully coupled and cooperative. It follows from Proposition 3.1 of [26] that there is a unique eigenvalue Λ_0 with positive eigenvector Φ_0 of

$$\left[\begin{array}{ll} \mathrm{L}u + Mu = \lambda\ I\ u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{38}$$

The eigenvalue Λ_0 is the first one and has multiplicity one. If we assume that $\Lambda_0 > 0$ one finds by Theorem 1.1 of [26] that $(\mathrm{L} + M)^{-1} : C_0(\bar{\Omega})^k \to C(\bar{\Omega})^k$ exists and satisfies $(\mathrm{L} + M)^{-1} > 0$.

Instead of (38) we consider the eigenvalue problem which has a nonnegative matrix B as a weight on the right hand side, namely

$$\left[\begin{array}{ll} \mathrm{L}u + Mu = \lambda\ B\ u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{39}$$

The eigenvalue problem with $B = -M$ is studied in [15]. See also [4].

Theorem 5.1 *Let* L+M *be cooperative and fully coupled. Suppose that* $\Lambda_0 > 0$. *Let the matrix* B *satisfy*

$$B_{ij}(\cdot) \in C_0(\bar{\Omega}) \quad \text{for all } i,j \in \{1,\ldots,k\},$$
$$B_{ij}(\cdot) \geq 0 \quad \text{for all } i,j \in \{1,\ldots,k\},$$
$$B \neq O.$$

Then there exists an eigenvalue $\Lambda_B(\mathrm{L} + M) > 0$ *of (39) such that*

1. *for* $\lambda < \Lambda_B(\mathrm{L} + M)$ *we have* L+$M - \lambda B$ *is invertible and* $(\mathrm{L} + M - \lambda B)^{-1} > 0$,
2. *the corresponding eigenspace is* $\{c\ \Phi_B; c \in I\!R\}$ *for some* $\Phi_B > 0$,
3. *(up to normalization)* Φ_B *is the only positive eigenfunction of (39),*
4. *there exist* $c_1, c_2 > 0$ *such that* $c_1\phi_0 \leq (\Phi_B)_i \leq c_2\phi_0$ *for all* $i \in \{1,\ldots,k\}$.

Remark 1: A converse also holds. Let L,M and B be as above. If $\Lambda_B(\mathrm{L} + M) > 0$ holds, then $\Lambda_0 > 0$.

Remark 2: Except for the estimate in *4* we do not need that $\partial\Omega$ is $C^{1,1}$. An exterior cone condition is sufficient to obtain a first eigenvalue with a unique positive eigenfunction. In the next prove we will not use the $C^{1,1}$ regularity of the boundary. Using $C^{1,1}$ some steps in the proof can be simplified by the strong maximum principle.

Proof. The main difficulty in the proof is the fact that B is not strictly positive. We just have $B \geq 0$. It can be solved as follows. Define

$$\Omega_j^B = \left\{ x \in \Omega; \sum_{i=1}^{k} B_{ij}(x) \neq 0 \right\},$$

$$\mathcal{C}_j^B = \left\{ f \in C\left(\overline{\Omega_j^B}\right); f(x) = 0 \text{ for } x \in \overline{\Omega_j^B} \cap \partial\Omega \right\},$$

$$\mathcal{C}^B = \mathcal{C}_1^B \times \cdots \times \mathcal{C}_k^B$$

The sets Ω_j^B are open subsets of Ω. $\mathcal{C}_j^B$ and $\mathcal{C}^B$ are Banach lattices. Denote

$$R_B : C(\bar{\Omega})^k \to \mathcal{C}^B \quad \text{the restriction,}$$

$$E_B : \mathcal{C}^B \to C(\bar{\Omega})^k \quad \text{the extension by zero.}$$

The operator $B \circ E_B : \mathcal{C}^B \to C(\bar{\Omega})^k$ is strictly positive: if $f \in \mathcal{C}^B$ satisfies $0 \leq f \neq 0$, then $0 \leq B \circ E_B f \neq 0$. The operator $B \circ E_B$ is continuous. Instead of the eigenvalue problem $\Phi = \lambda (\mathrm{L} + M)^{-1} B\Phi$ in $C(\bar{\Omega})^k$, we consider

$$\tilde{\Phi} = \lambda\ R_B\ (\mathrm{L} + M)^{-1} (B \circ E_B)\ \tilde{\Phi} \quad \text{in } \mathcal{C}^B. \tag{40}$$

By Lemma 1.4 of [26] the operator $(\mathrm{L} + M)^{-1} \in \mathsf{L}\left(C(\bar{\Omega})^n\right)$ is positive, irreducible and compact. One even has $(\mathrm{L} + M)^{-1} > 0$. Hence $R_B\ (\mathrm{L} + M)^{-1} (B \circ E_B) : \mathcal{C}^B \to \mathcal{C}^B$ is a positive, compact and irreducible operator. A theorem of De Pagter (Theorem 3 in [22]) shows that the spectral radius is strictly positive. By the Krein-Rutman Theorem ([18]) there exists a unique positive eigenvalue λ_B with a positive eigenfunction $\tilde{\Phi}_B$ in $\mathcal{C}^B$. One has $\lambda_B = \left(\rho\left(R_B\ (\mathrm{L} + M)^{-1} (B \circ E_B)\right)\right)^{-1}$. See also Theorem V.5.2 of [24] or the appendix of [26]. The eigenvalue/eigenfunction in $C(\bar{\Omega})^n$ is defined by

$$\begin{cases} \Lambda_B (\mathrm{L} + M) = \lambda_B \\ \Phi_B = \lambda_B\ (\mathrm{L} + M)^{-1} (B \circ E_B)\ \tilde{\Phi}_B. \end{cases}$$

Since $0 \neq \tilde{\Phi}_B \geq 0$ one finds $\Phi_B > 0$ and hence on a $C^{1,1}$-domain Hopf's boundary point Lemma implies *4*. It remains to show that Φ_B is unique up to normalization. If (Λ, Φ) with $\Phi \geq 0$ is an eigenvalue/function of (39), then $\Phi = \Lambda\ (\mathrm{L} + M)^{-1} B\ \Phi$ and hence $\Phi = 0$ or $\Phi > 0$. If $\Phi > 0$ holds, then $\tilde{\Phi} = R_B\Phi$ is a positive eigenfunction of (40). Since the positive eigenfunction of (40) is unique we have $\tilde{\Phi}_B = c\tilde{\Phi}$ for some $c > 0$. Hence $\Phi_B = \lambda_B\ (\mathrm{L} + M)^{-1} (B \circ E_B)\ \tilde{\Phi}_B = c\Phi$. □

5.2 Estimates for a vectorvalued Green function

Without loss of generality we may assume that $M_{ii} < 0$. (We may write $(\mathrm{L} + cI + M - cI)^{-1}$ instead of $(\mathrm{L} + M)^{-1}$.) For simplicity we define the positive matrix A by $A = -M$. The corresponding constant matrix $\hat{A}$ is defined by

$$\left(\hat{A}\right)_{ij} = \begin{cases} 1 & \text{if } \ 0 \neq A_{ij} \geq 0, \\ 0 & \text{if } \quad A_{ij} = 0. \end{cases} \tag{41}$$

We will assume that either $A_{ij} > 0$ on $\bar{\Omega}$ or $A_{ij} \equiv 0$.

Definition 5.2 *Let κ be the smallest number such that*

$$\left(\hat{A}^{\kappa}\right)_{ij} > 0 \quad \text{for all } i, j \in \{1, \ldots, k\}.$$

We call κ the coupling number of A (or M).

Since the system is fully coupled κ exists and is less than or equal to $k-1$.

Lemma 5.3 *Let $\mathrm{L}-A$ be cooperative and fully coupled. Suppose that $\Lambda_0 > 0$. Then there exist $c > 0$ such that, for $\kappa_n = \max\left(\kappa, \left[\frac{n+2}{2}\right]\right)$ we have*

$$c^{-1} \ \langle \Phi_A, f \rangle \ \Phi_A \leq \mathrm{L}^{-1} \left(A \ \mathrm{L}^{-1}\right)^{\kappa_n} f \leq c \ \langle \Phi_A, f \rangle \ \Phi_A \quad \text{for all } 0 \leq f \in C(\bar{\Omega})^n.$$

Here $\langle \Phi, f \rangle = \sum_{i=1}^{k} \int_{\Omega} \Phi_i(x) \ f_i(x) \ dx$ and Φ_A is the positive eigenfunction of (39) with $B = A = -M$.

Proof. Since $\kappa_n \geq \left[\frac{n+2}{2}\right]$ one finds by Lemma 4.3 that there exist $c_1, c_2 > 0$ such that

$$c_1 \ \phi_0(x) \int_{\Omega} \hat{A}^{\kappa_n} \ f(y) \ \phi_0(y) \ dy \leq \left(\mathrm{L}^{-1} \left(A \ \mathrm{L}^{-1}\right)^{\kappa_n} f\right)(x)$$

and

$$\left(\mathrm{L}^{-1} \left(A \ \mathrm{L}^{-1}\right)^{\kappa_n} f\right)(x) \leq c_2 \ \phi_0(x) \int_{\Omega} \hat{A}^{\kappa_n} \ f(y) \ \phi_0(y) \ dy$$

for all $x \in \Omega$ and $f \in C(\bar{\Omega})^+$. Since $\kappa_n \geq \kappa$ all entries of $\hat{A}^{\kappa_n}$ are strictly positive. Hence there exist $c_3, c_4 > 0$ such that for all $y \in \Omega$:

$$c_3 \ \mathbf{1} \cdot f(y) \ \mathbf{1} \leq \hat{A}^{\kappa_n} \ f(y) \leq c_4 \ \mathbf{1} \cdot f(y) \ \mathbf{1}.$$

By using Theorem 4.2 and

$$c_5 \ \phi_0(x) \ \mathbf{1} \leq \Phi_A(x) \leq c_6 \ \phi_0(x) \ \mathbf{1}$$

for some $c_5, c_6 > 0$ the proof concludes. □

Lemma 5.4 *Let L, A and κ_n be as in the previous Lemma. Let $\hat{A}$ be as in (41) and let $\mathcal{G}$ denote the Green function for the Laplacian with Dirichlet boundary condition. Then there exists $c > 0$ such that*

$$c^{-1} \sum_{m=0}^{\kappa_n} \hat{A}^m \ \mathcal{G}^{m+1} \ f \leq (\mathrm{L} - A)^{-1} f \leq c \sum_{m=0}^{\kappa_n} \hat{A}^m \ \mathcal{G}^{m+1} \ f \quad \text{for all } 0 \leq f \in C(\bar{\Omega})^k.$$

Remark 3: In the case that $0 \neq A_{ij} \geq 0$ but $A_{ij} \not> 0$ the first $k_n - 1$ terms have to be estimated in a more tedious way.

Proof. From Theorem 5.1 one finds that $\Lambda_A(\mathrm{L}-A)>0$. The operator $A\mathrm{L}^{-1}$ is positive and its spectral radius ρ satisfies

$$\rho=\rho\left(A\,\mathrm{L}^{-1}\right)=\left(\Lambda_A(\mathrm{L}-A)+1\right)^{-1}<1.$$

Hence we have

$$(\mathrm{L}-A)^{-1}=\mathrm{L}^{-1}\sum_{m=0}^{\infty}\left(A\,\mathrm{L}^{-1}\right)^m.$$

Using Lemma 5.3 and the positivity of A and L^{-1} we find for $m\geq\kappa_n$ and $0\leq f\in C(\bar\Omega)^k$ that

$$c^{-1}\,\rho^{-m+\kappa_n}\,\langle\Phi_A,f\rangle\;\Phi_A\leq\mathrm{L}^{-1}\left(A\mathrm{L}^{-1}\right)^m\;f\leq c\,\rho^{-m+\kappa_n}\,\langle\Phi_A,f\rangle\;\Phi_A. \tag{42}$$

Using $\mathrm{L}^{-1}A\Phi_A=\rho\Phi_A$, estimate (42), Theorem 4.1 and Theorem 4.2 it follows that for some $c_i>0$

$$\begin{aligned}(\mathrm{L}-A)^{-1}\;f\;&=\;\mathrm{L}^{-1}\sum_{m=0}^{\kappa_n-1}\left(A\,\mathrm{L}^{-1}\right)^m f\;+\mathrm{L}^{-1}\sum_{m=\kappa_n}^{\infty}\left(A\,\mathrm{L}^{-1}\right)^m f\quad\leq\\
&\leq\;\mathcal{G}\sum_{m=0}^{\kappa_n-1}(A\,\mathcal{G})^m f\;+c_1\sum_{m=\kappa_n}^{\infty}\rho^{-m+\kappa_n}\,\langle\Phi_A,f\rangle\;\Phi_A\quad\leq\\
&\leq\;c_2\,\mathcal{G}\sum_{m=0}^{\kappa_n-1}\left(\hat A\,\mathcal{G}\right)^m f\;+c_1\sum_{m=\kappa_n}^{\infty}\rho^{-m+\kappa_n}\,\langle\Phi_A,f\rangle\;\Phi_A\quad\leq\\
&\leq\;c_3\sum_{m=0}^{\kappa_n-1}\hat A^m\,\mathcal{G}^{m+1}f+c_4\,\frac{1}{1-\rho}\,\langle\Phi_A,f\rangle\;\Phi_A\;\leq\;c_5\sum_{m=0}^{\kappa_n}\hat A^m\,\mathcal{G}^{m+1}f.\end{aligned}$$

By using Lemma 4.3 one finds that there exists $c>0$ such that $\mathcal{G}\;\hat A\;\mathcal{G}<c\;\mathcal{G}\;A\;\mathcal{G}$ and the estimate from below follows similarly. □

6 Results for noncooperative systems

We consider

$$\left[\begin{array}{ll}\mathrm{L}u=Au-Bu+f & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{array}\right. \tag{43}$$

Now A and B are matrices of $C(\bar\Omega)$-functions with $A_{ij},B_{ij}\geq 0$ for $i\neq j$. We assume that $B_{ii}=A_{ii}=0$ (nonzero diagonal terms can be included in the elliptic operator L). Moreover, we assume that either $A_{ij}=0$ or $A_{ij}>0$ on $\bar\Omega$. We will also need a related cooperative system, namely

$$\left[\begin{array}{ll}\mathrm{L}u=Au+Bu+f & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{array}\right. \tag{44}$$

Note that L$-A$ being fully coupled implies that L$-A-B$ is fully coupled.

6.1 Existence

Theorem 6.1 *Let A, B be as above with* $\mathrm{L}-A$ *fully coupled. We assume that we have* $\Lambda_I(\mathrm{L}-A-B) > 0$. *Then for every* $f \in C_0(\bar{\Omega})^k$ *there exists a unique solution* u *of (43) in* $\left(C_0(\bar{\Omega}) \cap W^{p,2}(\Omega)\right)^k$, *with* $p \in [1,\infty)$, *and*

$$u = \sum_{m=0}^{\infty} \left(-(\mathrm{L}-A)^{-1} B\right)^m (\mathrm{L}-A)^{-1} f. \tag{45}$$

Remark 1: Note that $\Lambda_I(\mathrm{L}-A-B) > 0$ is a condition for the related cooperative system (44). The solution $\tilde{u}$ of (44) satisfies

$$\tilde{u} = \sum_{m=0}^{\infty} \left((\mathrm{L}-A)^{-1} B\right)^m (\mathrm{L}-A)^{-1} f. \tag{46}$$

Remark 2: Instead of assuming that $\partial\Omega \in C^{1,1}$ one may assume that $\partial\Omega$ is regular (see [13]) for the operators L_i. The theorem still holds with $u \in \left(C_0(\bar{\Omega}) \cap W^{p,2}_{loc}(\Omega)\right)^k$.

Proof. Since $\Lambda_I(\mathrm{L}-A-B) > 0$ we find $\Lambda_I(\mathrm{L}-A) > 0$ and $\Lambda_B(\mathrm{L}-A) > 1$. By Theorem 5.1 there exists a unique positive eigenfunction of (39). Since $(\mathrm{L}-A)^{-1} B$ is positive and compact, the Krein-Rutman Theorem implies that $\rho\left((\mathrm{L}-A)^{-1} B\right) = (\Lambda_B(\mathrm{L}-A))^{-1} < 1$. Hence the formulas in (43) and (44) are well defined. The solution in (43) is unique. Indeed, since $(\mathrm{L}-A)^{-1}$ is well defined

$$\left[\begin{array}{ll} (\mathrm{L}-A)\, u = -Bu + f & \text{in } \Omega \\ \qquad u = 0 & \text{on } \partial\Omega \end{array}\right.$$

is equivalent with

$$u = (\mathrm{L}-A)^{-1}(-Bu + f)$$

or

$$\left(I + (\mathrm{L}-A)^{-1} B\right) u = (\mathrm{L}-A)^{-1} f.$$

Since $\rho\left((\mathrm{L}-A)^{-1} B\right) < 1$ the left hand side can be inverted. Regularity follows from the regularizing property of $(\mathrm{L}-A)^{-1}$ since $u = (\mathrm{L}-A)^{-1} w$ where $w \in C(\bar{\Omega})^k$ is defined by $w = \sum_{m=0}^{\infty} \left(-B(\mathrm{L}-A)^{-1}\right)^m f$. □

6.2 Motivation for restricted positivity

Another way of writing the solution of (43) is

$$u = \sum_{m=0}^{\infty} \left((\mathrm{L} - A)^{-1} B \right)^{2m} \left(I - (\mathrm{L} - A)^{-1} B \right) (\mathrm{L} - A)^{-1} f.$$

The operators $\sum_{m=0}^{\infty} \left((\mathrm{L} - A)^{-1} B \right)^{2m}$ and $(\mathrm{L} - A)^{-1}$ are positive. But the third operator, $\left(I - (\mathrm{L} - A)^{-1} B \right)$, is not positive, not even when the coefficients of B are small. However, in some cases the operator $\left(I - (\mathrm{L} - A)^{-1} B \right) (\mathrm{L} - A)^{-1}$ will preserve a subcone of the positive cone for small B. It motivates the following question.

- *Which subcones of the positive cone are preserved by the operator*

$$T_\varepsilon = \left(I - \varepsilon (\mathrm{L} - A)^{-1} B \right) (\mathrm{L} - A)^{-1} : C(\bar{\Omega})^k \to C(\bar{\Omega})^k \tag{47}$$

for small $\varepsilon > 0$ *?*

If we set $u = T_\varepsilon f$ and $v = T_0 f$ we find the following system:

$$\begin{cases} (\mathrm{L} - A)\, u = f - \varepsilon B v & \text{in } \Omega, \\ (\mathrm{L} - A)\, v = f & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{48}$$

In the one equation case it is shown (see [25], [5], [6] and [26]) that $\left(1 - \varepsilon (-\Delta)^{-1} \right) (-\Delta)^{-1}$ is positive for sufficiently small ε. This operator corresponds with a system as in (48), namely:

$$\begin{cases} -\Delta u = f - \varepsilon\, v & \text{in } \Omega, \\ -\Delta v = f & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Other related questions are

- *For which subcone* $K \subset \left(C(\bar{\Omega})^k \right)^+$ *does one find* $T_\varepsilon K \subset \left(C(\bar{\Omega})^k \right)^+$ *?*
- *If* $K = \left(C(\bar{\Omega})^k \right)^+$ *which components satisfy* $(T_\varepsilon K)_i \geq 0$ *?*

Note that $T_1 f \geq 0$ implies that the solution u of (43) satisfies $u \geq 0$.

6.3 Restricted positivity

Which subcone of the positive cone will be preserved depends merely on the way how the system is coupled, that is: on the matrices A and B. We need some technical conditions. To compare the coupling of A and B we need the following.

Definition 6.2 *Let $\hat{A}$ and $\hat{B}$ be as in (41). We define the matrix $\hat{B}_A \in \mathbb{N}^{k\times k}$ by*

$$\left(\hat{B}_A\right)_{ij} = 1 + \min\left\{ m \in \mathbb{N};\ \left(\sum_{p=0}^{m} \hat{A}^p \hat{B} \hat{A}^{m-p} \mathbf{e}^j\right)_i \neq 0 \right\}.$$

Here $\mathbf{e}^j$ is the j^{th} unit vector in $\mathbb{R}^k$.

Remark 3: Let Γ_A and Γ_B denote the directed graphs corresponding with the coupling of $\hat{A}$ respectively $\hat{B}$ ($1, \ldots, k$ are the nodes, there is an arc in Γ_A from j to i iff $\hat{A}_{ij} \neq 0$). The number in $\left(\hat{B}_A\right)_{ij}$ denotes the length of (= number of directed arcs in) the shortest path in $\Gamma_A \cup \Gamma_B$ from j to i that uses exactly one arc of Γ_B.

We will also use

$$\begin{cases} P_i : C(\bar{\Omega})^k \to C(\bar{\Omega}), \\ P_i u = u_i, \end{cases}$$

and its right inverse

$$\begin{cases} E_i : C(\bar{\Omega}) \to C(\bar{\Omega})^k, \\ E_i u = (0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{th}}\ \text{entry}}}{u}, 0, \ldots, 0)\,. \end{cases}$$

Theorem 6.3 *Let A, B be as above with $\mathrm{L}-A$ fully coupled. Moreover, we assume that $\Lambda_I(\mathrm{L} - A - B) > 0$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ the following holds. Let $i, j \in \{1, \ldots, k\}$.*
If $i = j$ or

$$\left(\hat{B}_A\right)_{ij} \geq \min\left\{\left[\frac{n+2}{2}\right], \left(\hat{A}_A\right)_{ij}\right\} \tag{49}$$

then $P_i \circ T_\varepsilon \circ E_j > 0$.

If we set

$$\chi = \left\{ i; \left(\hat{B}_A\right)_{ij} \geq \min\left\{\left[\tfrac{n+2}{2}\right], \left(\hat{A}_A\right)_{ij}\right\} \text{ for all } j \neq i \right\}$$
$$\varsigma = \left\{ j; \left(\hat{B}_A\right)_{ij} \geq \min\left\{\left[\tfrac{n+2}{2}\right], \left(\hat{A}_A\right)_{ij}\right\} \text{ for all } i \neq j \right\}$$

the main result follows directly from the previous theorem.

Theorem 6.4 *Let A, B be as above with $\mathrm{L}-A$ fully coupled and $\Lambda_I(\mathrm{L}-A-B)>0$. Let $\varepsilon_0>0$ be as above. Then for all $\varepsilon\in[0,\varepsilon_0]$ we have the following. Let $f\in C(\bar{\Omega})^k$.*

1. *If $f\geq 0$, then $(T_\varepsilon f)_i\geq 0$ for all $i\in\chi$.*
2. *If $f\geq 0$ and $f_j=0$ for all $j\notin\varsigma$, then $T_\varepsilon f\geq 0$.*

Examples. The entries where (49) is satisfied contain $\cdot$ and we put a $\star$ if (49) is not satisfied.

1. Let

$$A=\hat{A}=\begin{pmatrix}1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&1\\0&0&0&1&1\\1&0&0&0&1\end{pmatrix}\text{ and }B=\hat{B}=\begin{pmatrix}0&0&1&0&0\\1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\end{pmatrix}.$$

Then

$$\hat{A}_A=\begin{pmatrix}1&1&2&3&4\\4&1&1&2&3\\3&4&1&1&2\\2&3&4&1&1\\1&2&3&4&1\end{pmatrix}\text{ and }\hat{B}_A=\begin{pmatrix}2&3&1&2&3\\1&2&3&4&5\\5&6&2&3&4\\4&5&1&2&3\\3&4&2&3&4\end{pmatrix}$$

and the critical entries, respectively the index sets χ and ς appear as follows:

for $n=2,3$

$$\begin{pmatrix}\cdot&\cdot&\star&\cdot&\cdot\\\star&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\star&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$\chi=\{2,4,5\}$
$\varsigma=\{3,5\}$

for $n=4,5$

$$\begin{pmatrix}\cdot&\cdot&\star&\star&\cdot\\\star&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\star&\cdot&\cdot\\\cdot&\cdot&\star&\cdot&\cdot\end{pmatrix}$$

$\chi=\{2,5\}$
$\varsigma=\{3\}$

for $n\geq 6$

$$\begin{pmatrix}\cdot&\cdot&\star&\star&\star\\\star&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\star&\cdot&\cdot\\\cdot&\cdot&\star&\star&\cdot\end{pmatrix}$$

$\chi=\{2\}$
$\varsigma=\{3\}$.

2. Let A be the same and replace B by

$$B=\hat{B}=\begin{pmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&0&0&0&0\\0&1&0&0&0\end{pmatrix}.\text{ Then for }n=2:\begin{pmatrix}\cdot&\cdot&\star&\cdot&\cdot\\\cdot&\cdot&\cdot&\star&\cdot\\\cdot&\cdot&\cdot&\cdot&\star\\\star&\cdot&\cdot&\cdot&\cdot\\\cdot&\star&\cdot&\cdot&\cdot\end{pmatrix}$$

For all $n\geq 2$ we get $\chi=\varsigma=\emptyset$.

Remark 4: The dots show that the corresponding coupling is positive for $\varepsilon > 0$ but small enough. The theorem does not give an actual estimate for ε_0. For a particular system such estimates can be obtained. See the last section where the system from McKenna and Walter is studied.

Proof of Theorem 6.3. The condition in (49) shows that there is $s_0 > 0$ such that for $s \in [0, s_0]$ one finds

$$\left(\left(\hat{A}^{m+1} - s \sum_{p=0}^{m} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p}\right) \mathbf{e}^j\right)_i \geq 0 \quad \text{for } m \leq \left[\frac{n+2}{2}\right]. \tag{50}$$

There also is $s_1 > 0$ such that for $s \in [0, s_1]$ one finds

$$\left(\left(\hat{A}^{\kappa_n} - s \sum_{p=0}^{m} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p}\right) \mathbf{e}^j\right)_i \geq 0 \quad \text{for } m > \left[\frac{n+2}{2}\right]. \tag{51}$$

From (50) it follows that for small s

$$\left(\sum_{m=0}^{\left[\frac{n+2}{2}\right]} \left(\hat{A}^{m+1} - s \sum_{p=0}^{m} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p}\right) \mathcal{G}^{m+2} \ E_j f\right)_i \geq 0 \tag{52}$$

and from (51) and Theorem 4.2 it follows that for small s

$$\left(\left(\hat{A}^{\kappa_n}\mathcal{G}^{\kappa_n+1} - s \sum_{m=\left[\frac{n+2}{2}\right]+1}^{\kappa_n-1} \ \sum_{p=0}^{m} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p}\mathcal{G}^{m+2}\right) E_j f\right)_i \geq 0 \tag{53}$$

Since $\mathcal{G}$ does not mix the components we find for all $0 \leq f \in C(\bar{\Omega})$ that we have for small s

$$\left(\sum_{m=0}^{\kappa_n-1} \left(\hat{A}^{m+1} - s \sum_{p=0}^{m} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p}\right) \mathcal{G}^{m+2} \ E_j f\right)_i \geq 0. \tag{54}$$

If $m+1 \geq \left[\frac{n+2}{2}\right] + 1$ we find by Theorem 4.2 and Lemma 5.3 that there is $c_1 > 0$ such that for all $0 \leq v \in C(\bar{\Omega})^k$ we have

$$c_1 \hat{A}^p \ \hat{B} \ \hat{A}^{m-p-1} \ \mathcal{G}^{m+1} v \geq \hat{A}^{\kappa_n}\mathcal{G}^{\kappa_n+1} v. \tag{55}$$

Hence there is $s_1 > 0$ such that for $s \in [0, s_1]$ and $0 \leq f \in C(\bar{\Omega})$:

$$\left(\left(\sum_{m=-1}^{\kappa_n-1} \hat{A}^{m+1} \ \mathcal{G}^{m+2} - s \sum_{m=0}^{2\kappa_n} \left(\sum_{p=\max(0,m-\kappa_n)}^{\min(m,\kappa_n)} \hat{A}^p \hat{B} \ \hat{A}^{m-p}\right) \mathcal{G}^{m+2}\right) E_j f\right)_i \geq 0. \tag{56}$$

By Lemma 5.4 one finds that there are $c_2, c_3 > 0$ such that for all $0 \le v \in C(\bar{\Omega})^k$ we have

$$T_\varepsilon v \ge c_3 \sum_{m=0}^{\kappa_n} \hat{A}^m \ \mathcal{G}^{m+1} v - c_2 \ \varepsilon \ \sum_{p=0}^{\kappa_n} \hat{A}^p \mathcal{G}^{p+1} \ \hat{B} \ \sum_{q=0}^{\kappa_n} \hat{A}^q \ \mathcal{G}^{q+1} v =$$

$$= c_3 \left(\sum_{m=-1}^{\kappa_n - 1} \hat{A}^{m+1} \ \mathcal{G}^{m+2} - \frac{c_2}{c_3} \ \varepsilon \sum_{m=0}^{2\kappa_n} \left(\sum_{p=\max(0,m-\kappa_n)}^{\min(m,\kappa_n)} \hat{A}^p \ \hat{B} \ \hat{A}^{m-p} \right) \mathcal{G}^{m+2} \right) v. \tag{57}$$

Combining (56) with (57) one finds that $P_i \circ T_\varepsilon \circ E_j$ is positive for small ε. □

7 Application

Walter an McKenna proposed the problem

$$\left[\begin{array}{ll} \Delta^2 u + b(\cdot) \ u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{58}$$

They studied (58) in the one dimensional case in relation with a problem appearing for nonlinear oscillations of a suspension bridge. We will consider the higher dimensional case.

Setting $v = -\Delta u$ one obtains with $b(x) > 0$ for some $x \in \Omega$ the noncooperative system

$$\left[\begin{array}{ll} -\Delta \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 0 & -b(\cdot) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} f \\ 0 \end{pmatrix} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{59}$$

We will write $a(\cdot) = -b(\cdot)$.

$a \ge 0$. Then (and only then) the system in (59) is cooperative. The system is positivity preserving if and only if there is a positive strict supersolution (see [12], [26]). A sufficient condition for $(\phi_0, t\phi_0)$ to be such a solution is $a(x) < \lambda_0^2$. Another way of getting a supersolution is $(\phi_a, t\phi_a)$ with $\lambda_a^2 < a(x)^{-1}$, where (λ_a, ϕ_a) is the eigenfunction of $-\Delta\phi = a\phi$ in Ω, $\phi = 0$ on $\partial\Omega$.

$a \le 0$. We cannot apply Theorem 6.3 straightforwardly since the cooperative part is not fully coupled. This only gives a minor difficulty. Trying to solve (58,59) for u we obtain

$$\left(I + (-\Delta)^{-2} b(\cdot) \right) u = (-\Delta)^{-1} f.$$

Writing B for the multiplication by b and assuming $\rho(\mathcal{G}^2 B) < 1$ we obtain

$$u = \sum_{m=1}^{\infty} \left(\mathcal{G}^2 B \right)^{2k} \left(I - \mathcal{G}^2 B \right) \mathcal{G} \ f.$$

Since $0 \le B$ and $0 \le \mathcal{G}^2 B \le \mathcal{G}^2 \|b\|_\infty$ we find that $b(x)^2 < \lambda_0$ is a sufficient condition for $\rho(\mathcal{G}^2 B) < 1$. For positivity of $(I - \mathcal{G}^2 B)\mathcal{G}$ note that

$$\left(I - \mathcal{G}^2 B\right)\mathcal{G} \; = (I - \sigma\mathcal{G})\,\mathcal{G} + \sigma\mathcal{G}\left(I - \frac{1}{\sigma}\mathcal{G}B\right)\mathcal{G}\;. \tag{60}$$

Let λ_c denote the largest constant such that $(I - \lambda\mathcal{G})\,\mathcal{G}$ is positive. Then with $\sigma = \sqrt{\|b\|_\infty}$ we find that the operator in (60) is positive for $\|b\|_\infty \le \lambda_c^2$.

a changes sign. We write $a = a_+ - a_-$ and set $\sigma = \sqrt{\|a\|_\infty}$. The system that we use is

$$\left[\begin{array}{ll} -\Delta \times I_2\ w = \begin{pmatrix} 0 & a_+/\sigma \\ \sigma & 0 \end{pmatrix} w - \begin{pmatrix} 0 & a_-/\sigma \\ 0 & 0 \end{pmatrix} w + g & \text{in } \Omega, \\ \qquad w = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{61}$$

with $w = \binom{v/\sigma}{u}$ and $g = \binom{f/\sigma}{0}$. With our previous notation $A = \binom{0\ \ a_+/\sigma}{\sigma\ \ 0}$ and $B = \binom{0\ \ a_-/\sigma}{0\ \ 0}$. The operator $-\Delta \times I_2 - A$ can be inverted if $\|a_+\|_\infty < \lambda_0^2$. In order to apply Theorem 6.1 we need a positive strict supersolution for $-\Delta \times I_2 - A - B$. The couple (ϕ_0, ϕ_0) is such a supersolution if the following inequalities are satisfied

$$\left\{\begin{array}{l} \lambda_0\phi_0 > \dfrac{a_+}{\sigma}\ \phi_0 + \dfrac{a_-}{\sigma}\ \phi_0, \\ \qquad \lambda_0\phi_0 > \sigma\ \phi_0. \end{array}\right.$$

Since $\|a\|_\infty = \|a_+ - a_-\|_\infty = \max(\|a_+\|_\infty, \|a_-\|_\infty) = \|a_+ + a_-\|_\infty$ a sufficient condition for existence is $\|a\|_\infty < \lambda_0^2$. Now using

$$(-\Delta \times I_2 - A + B)^{-1} =$$

$$= \sum_{m=0}^{\infty} \left((-\Delta \times I_2 - A)^{-1} B\right)^{2m} \left(I_2 - (-\Delta \times I_2 - A)^{-1} B\right)(-\Delta \times I_2 - A)^{-1}$$

it is sufficient to show positivity of $\left(I_2 - (-\Delta \times I_2 - A)^{-1} B\right)(-\Delta \times I_2 - A)^{-1}$. As above we have

$$(-\Delta \times I_2 - A)^{-1} = \sum_{m=0}^{\infty} (\mathcal{G}\ A)^m\, \mathcal{G} = \sum_{m=0}^{\infty} (\mathcal{G}A)^{2m}\ \mathcal{G}\ \ (I_2 + A\ \mathcal{G}) =$$

$$= \sum_{m=0}^{\infty} \begin{pmatrix} \mathcal{G}a_+\mathcal{G} & 0 \\ 0 & \mathcal{G}^2 a_+ \end{pmatrix}^{2m} \begin{pmatrix} \mathcal{G} & 0 \\ 0 & \mathcal{G} \end{pmatrix} \begin{pmatrix} 1 & a_+/\sigma\ \mathcal{G} \\ \sigma\ \mathcal{G} & 1 \end{pmatrix}.$$

Or similarly

$$(-\Delta \times I_2 - A)^{-1} = \begin{pmatrix} 1 & \mathcal{G}\ a_+/\sigma \\ \sigma\ \mathcal{G} & 1 \end{pmatrix} \sum_{m=0}^{\infty} \begin{pmatrix} \mathcal{G}a_+\mathcal{G} & 0 \\ 0 & \mathcal{G}^2 a_+ \end{pmatrix}^{2m} \begin{pmatrix} \mathcal{G} & 0 \\ 0 & \mathcal{G} \end{pmatrix}.$$

With some computations we get

$$\left(I_2 - (-\Delta \times I_2 - A)^{-1} B\right)(-\Delta \times I_2 - A)^{-1} \begin{pmatrix} f/\sigma \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ \sigma\mathcal{G} \end{pmatrix} \sum_{m=0}^{\infty} (\mathcal{G}a_+\mathcal{G})^m \left(\mathcal{G} - \mathcal{G}a_-\mathcal{G}\sum_{i=0}^{\infty} (\mathcal{G}a_+\mathcal{G})^i \mathcal{G}\right) f/\sigma. \tag{62}$$

It shows that is sufficient to have $\mathcal{G}\left(I - a_-\mathcal{G}\sum_{i=0}^{\infty} (\mathcal{G}a_+\mathcal{G})^i \mathcal{G}\right)$ positive. Using that $\mathcal{G}^2 \leq \lambda_c^{-1}\mathcal{G}$ we have if $\|a_+\|_\infty < \lambda_c^2$ that

$$\mathcal{G}\sum_{i=0}^{\infty} (\mathcal{G}a_+\mathcal{G})^i \mathcal{G} \leq \sum_{i=0}^{\infty} (\|a_+\|_\infty)^i \ \mathcal{G}^{2i+2} \leq$$

$$\leq \sum_{i=0}^{\infty} (\|a_+\|_\infty)^i \ \lambda_c^{-2i} \ \mathcal{G}^2 = \frac{\lambda_c^2}{\lambda_c^2 - \|a_+\|_\infty}\mathcal{G}^2. \tag{63}$$

Hence

$$\mathcal{G}\left(I - a_-\mathcal{G}\sum_{i=0}^{\infty} (\mathcal{G}a_+\mathcal{G})^i \mathcal{G}\right) \geq$$

$$\geq \mathcal{G}\left(I - \|a_-\|_\infty \frac{\lambda_c^2}{\lambda_c^2 - \|a_+\|_\infty}\mathcal{G}^2\right) \geq$$

$$\geq \frac{\lambda_c^2 - \|a_+\|_\infty - \|a_-\|_\infty}{\lambda_c^2 - \|a_+\|_\infty}\mathcal{G} \tag{64}$$

and we find that $f \geq 0$ implies $u \geq 0$ if $\|a_+\|_\infty + \|a_-\|_\infty < \lambda_c^2$. The condition is the same as for negative a.

We summarize:

Theorem 7.1 *Let u be a solution of (58). Let λ_c be the largest constant such that $\mathcal{G} - \lambda\mathcal{G}^2 > 0$. If*

1. *$a \geq 0$ and $\|a\|_\infty < \lambda_1^2$, or*
2. *$a \leq 0$ and $\|a\|_\infty < \lambda_c^2$, or*
3. *a changes sign and $\|a_+\|_\infty + \|a_-\|_\infty < \lambda_c^2$,*

then u is unique and $f \geq 0$ implies $u \geq 0$.

Remark 1: From the closely related Barta inequality (see [2]) one finds that the first eigenvalue satisfies

$$\lambda_0 = \sup_{f \in K} \inf_{x \in \Omega} \frac{(\mathcal{G}f)(x)}{(\mathcal{G}^2 f)(x)},$$

where $K = \{f \in C(\bar{\Omega}); 0 \leq f \neq 0\}$. Note that the anti-eigenvalue λ_c satisfies

$$\lambda_c = \inf_{f \in K} \inf_{x \in \Omega} \frac{(\mathcal{G}f)(x)}{(\mathcal{G}^2 f)(x)}.$$

Remark 2: Instead of λ_c^2 one may use $\lambda_{cc} > \lambda_c^2$, where λ_{cc} denotes the largest constant such that $\mathcal{G} - \lambda \mathcal{G}^3$ is positive. Similar arguments can be used to close some of the gap between *1* and *3* in the theorem. Let λ_{c^m} the largest constant such that $\mathcal{G} - \lambda \mathcal{G}^{1+m}$ is positive. Then $\sqrt[m]{\lambda_{c^m}} \to \lambda_1$ as $m \to \infty$ and a more careful estimate would replace (63) for any $\varepsilon > 0$ with

$$\mathcal{G} \sum_{i=0}^{\infty} (\mathcal{G} a_+ \mathcal{G})^i \, \mathcal{G} \leq C(\varepsilon) \ \frac{\lambda_1^2}{\lambda_1^2 - (1+\varepsilon) \|a_+\|_\infty} \mathcal{G}^2.$$

The estimate in *3* is then replaced by

$$(1+\varepsilon) \frac{\|a_+\|_\infty}{\lambda_1^2} + C(\varepsilon) \frac{\|a_-\|_\infty}{\lambda_c^2} < 1.$$

Remark 3: Explicit values for λ_c in the radial symmetric case are found in [5]. For general domains estimates of λ_c can be found by using a relation with conditional Brownian motion. The constant $(\lambda_c)^{-1}$ equals $\sup \{\tau(x,y); x, y \in \Omega\}$ where $\tau(x,y)$ denotes the expected lifetime of Brownian motion starting at x, killed on $\partial\Omega$ and conditioned to converge to y. See [26]. Optimal estimates in two dimensional domains can be found in [14].

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