Addendum without Zorn to [DS]

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Consider

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω a bounded regular domain and $f \in C\left(\mathbb{R} \times \bar{\Omega}\right)$. Suppose that $\underline{u} \leq \overline{u}$ are respectively a sub- and a supersolution in appropriate sense. Denote by the set $\mathcal{F}$ all solutions in $[\underline{u}, \overline{u}]$.

Lemma 1 *If $\overline{u}$ is a supersolution and $\underline{u} \leq \overline{u}$ is a subsolution then there exists a solution u in between:*

$$\underline{u} \leq u \leq \overline{u}.$$

Lemma 2 *If u_1 and u_2 are supersolutions, then $u(x) = \min(u_1(x), u_2(x))$ is a supersolution.*

Lemma 3 *The set $\mathcal{F}$ is equicontinuous. The infimum of a totally ordered subset in $\mathcal{F}$ is a solution, that is every totally ordered subset of $\mathcal{F}$ has a minimum.*

Proposition 4 *There is a minimal solution $u_{\min}$ in $[\underline{u}, \overline{u}]$: for every solution $u \in [\underline{u}, \overline{u}]$ it holds that*

$$\underline{u} \leq u_{\min} \leq u \leq \overline{u}.$$

Proof. By Lemma 21 $\mathcal{F}$ is nonempty. For every $x \in \Omega$ we define $U(x) = \inf\{u_\lambda(x); u_\lambda \in \mathcal{F}\}$. We will show that $U = u_{\min}$, the minimal solution.

If $\mathcal{F}$ is finite, Lemma 2 implies that the function $\overline{u}^*(x) = \min\{u(x); u \in \mathcal{F}\}$ is a supersolution and Lemma 1 implies the existence of a solution $u_1 \in [\underline{u}, \overline{u}^*]$, that is $u_1 = \overline{u}^* \in \mathcal{F}$ is minimal.

Now suppose that $\mathcal{F}$ is not finite. First we will show that for every $x \in \Omega$ there is $u^x \in \mathcal{F}$ with $u^x(x) = U(x)$. Let us fix $x \in \Omega$.

1. By the definition of U there exists $\{u_i^x\}_{i \in \mathbb{N}} \subset \mathcal{F}$ such that $\lim_{i\to\infty} u_i^x(x) = U(x)$.

2. Define a new sequence $\{\widetilde{u}_i^x\}_{i \in \mathbb{N}} \subset \mathcal{F}$ as follows by iteration:

   i. $\widetilde{u}_0^x = u_0$ for some $u_0 \in \mathcal{F}$,
   ii. let $\widetilde{u}_{i+1}^x$ be a solution in $\left[\underline{u}, \min\left(\widetilde{u}_i^x, u_{i+1}^x\right)\right]$.

   Note that Lemma 2 implies that $\min\left(\widetilde{u}_i^x, u_{i+1}^x\right)$ is a supersolution and that Lemma 1 gives the existence of a solution $\widetilde{u}_{i+1}^x$. The set $\{\widetilde{u}_i^x\}_{i \in \mathbb{N}}$ is a totally ordered set in $\mathcal{F}$ and hence by Lemma 3 its infimum is a solution. Let us denote this function by $\widetilde{u}^x$.

3. Next let $\{x_i\}_{i \in \mathbb{N}}$ be a countable dense subset of Ω and define a sequence $\{u_i\}_{i \in \mathbb{N}}$ of solutions as follows:

   i. $u_0 = u_0$ for some $u_0 \in \mathcal{F}$,
   ii. let u_{i+1} be a solution in $[\underline{u}, \min(u_i, \widetilde{u}^{x_i})]$.

   Proceeding as for the previous sequence one finds that $\{u_i\}_{i \in \mathbb{N}}$ is a totally ordered set and that $u_\infty(x) = \lim_{i\to\infty} u_i(x)$ lies in $\mathcal{F}$.

Since $u_\infty(x_i) \leq U(x_i)$ for the dense set $\{x_i\}_{i \in \mathbb{N}}$ and $U(x) \leq u_\infty(x)$ for all $x \in \Omega$ we find that $U = u_\infty = u_{\min}$. ∎

References

[DS] E.N. Dancer, G. Sweers, On the existence of a maximal weak solution for a semilinear elliptic equation, Differential and Integral Equations 2 (1989), 533-540.