Partitions and quasimodular forms:

Variations on the Bloch-Okounkov theorem

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March 17, 2021: Vanderbilt University

The q-bracket

$\lambda = (\lambda_1, \lambda_2, \ldots)$
$|\lambda| = \sum_i \lambda_i$

For $f : \mathcal{P} \to \overline{\mathbb{Q}}$, we let

partitions of integers

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \in \overline{\mathbb{Q}}[[q]]$$

$\prod_{n \geq 1} (1 - q^n) = q^{-1/24} \eta$

Example $f(\lambda) = |\lambda| - \frac{1}{24}$.

$$\langle f \rangle_q = -\frac{1}{24} + q \frac{\partial}{\partial q} \log \Big(\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \Big) = -\frac{1}{24} + q \frac{\partial}{\partial q} \log \eta^{-1} q^{1/24} = E_2$$

$-\frac{1}{24} + \sum_{m, r \geq 1} q^{mr}$

The generating series of shifted symmetric functions

Let $\lambda \in \mathcal{P}$, $\underline{e}(x) = e^{2\pi i x}$, $\tau \in \mathfrak{h}$, $q = \underline{e}(\tau)$ and $z, z_1, \ldots, z_n \in \mathbb{C}$.

$$W_\lambda(z) := \sum_{i=1}^{\infty} \underline{e}((\lambda_i - i + \tfrac{1}{2})z) \qquad (\operatorname{Im} z < 0)$$

n-point function

$$\Theta(z) := \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} (-1)^{\lfloor \nu \rfloor} \underline{e}(\nu z)\, q^{\nu^2/2}$$

Writing $F_n(z_1, \ldots, z_n) := \langle W(z_1) \cdots W(z_n) \rangle_q =: \sum_{\sigma \in \mathfrak{S}_n} V_n(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$, we have

Thm (Bloch-Okounkov)

derivative

$$\sum_{m=0}^{n} \frac{(-1)^{n-m}}{(n-m)!} \Theta^{(n-m)}(z_1 + \cdots + z_m)\, V_m(z_1, \ldots, z_m) = 0.$$

Example $V_1(z) = \dfrac{\Theta'(0)}{\Theta(z)}$, $V_2(z_1, z_2) = \dfrac{\Theta'(0)}{\Theta(z_1+z_2)} \dfrac{\Theta'(z_1)}{\Theta(z_1)}$.

Proof sketch ($n=1$, following Zagier)

For $X \subseteq \mathbb{Q}$, write $W_X(z) = \sum_{x \in X} \underline{e}(xz)$ (formally),

so that $e^{\nu z} W_X(z) = W_{X+\nu}(z)$ $\quad (\lambda \in \mathcal{P}, \nu \in \mathbb{Q})$ (1)

For $\lambda \in \mathcal{P}$, let $X_\lambda = \{\lambda_i - i + \frac{1}{2}\}_{i=1}^{\infty}$, so that $W_\lambda(z) = W_{X_\lambda}(z)$ (2)

In particular, $[z^0] W_{X_\lambda + \nu}(z) = \nu$, $[z^1] W_{X_\lambda+\nu}(z) = |\lambda| + \frac{\nu^2}{2} - \frac{1}{24}$. (3)

Letting $X^\circledast = \begin{cases} X \setminus \{0\} & 0 \in X \\ X \cup \{0\} & 0 \notin X, \end{cases}$ we have $W_{X^\circledast}(z) = W_X(z) \pm 1$ (4)

Hence,

$$\frac{\Theta(z) \langle V_1(z) \rangle_q}{\eta} = \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} \sum_{\lambda \in \mathcal{P}} (-1)^{\lfloor \nu \rfloor} e^{\nu z} W_{X_\lambda}(z)\, q^{|\lambda| + \nu^2 - \frac{1}{24}}$$

$$= \sum_{\nu \in \mathbb{Z}+\frac{1}{2}} \sum_{\lambda \in \mathcal{P}} (-1)^{\lfloor \nu \rfloor} W_{X_\lambda + \nu}(z)\, q^{|\lambda| + \frac{\nu^2}{2} - \frac{1}{24}}$$

independent of z!

Overview

$$\begin{tikzcd}
\mathbb{Q}[W(z_i)] \arrow[r, "\langle\ \rangle_q"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{\mathcal{J}} \arrow[d, "\text{``Taylor around } z=0\text{''}"] \\
\Lambda^{*} \arrow[r, "\langle\ \rangle_q"] & \widetilde{M}
\end{tikzcd}$$

strictly meromorphic quasi-Jacobi forms, e.g. $\frac{\Theta'(0)}{\Theta(z)}$, F_n and their derivatives

Key property: Taylor coefficients at rational points are quasimodular

quasimodular forms $\simeq \mathbb{Q}[E_2, E_4, E_6]$, where $E_k = \frac{-B_k}{2k} + \sum_{m,r\geq 1} m^{k-1} q^{mr}$

(B_k = kth Bernoulli number)

shifted symmetric functions

$\Lambda^{*} := \mathbb{Q}[Q_2, Q_3, Q_4, \ldots]$ with

$$Q_k(\lambda) := [z^{k-1}]\, W_\lambda(z)$$

$$= \beta_k + \frac{1}{(k-1)!} \sum_i (\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1}$$

$$\left(\sum_i \beta_i z^{i-1} = \frac{1}{2\sinh(\frac{z}{2})} \right).$$

Examples of functions on partitions

(i) shifted symmetric functions Q_k [Bloch-Okounkov, '00]

$\mathrm{wt}\, Q_k = k$ [Okounkov-Pandharipande, '02]: Hurwitz/Gromov-Witten invariants

Cor For all $f \in \Lambda^*_k$ we have $f \in \widetilde{M}_k$.

(ii) p-adic analogues [Griffin-Jameson-Trebat-Leder, '16]

[Eskin-Okounkov, '06] [Engel, '17]: Orbifold Hurwitz theory

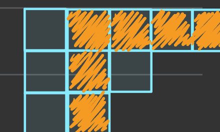
$$Q_k^{(p)}(\lambda) := \beta_k\left(1-\frac{1}{p}\right) + \frac{1}{(k-1)!} \sum_{(2\lambda_i - 2i+1, p)=1} \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1} \right)$$

Examples of functions on partitions

(iii) hook-length moments / t-hook functions [Bringmann-Ono-Wagner, '20]

[Chen-Möller-Zagier, '16]: Dynamics of flat surfaces

$$H_k^{(t)}(\lambda) := -\frac{B_k}{2k}t^k + \sum_{\substack{\xi \in Y_\lambda \\ t \mid h(\xi)}} h(\xi)^{k-2}.$$



hook-length of cell ξ

(iv) moment functions [Zagier, '16] [vI, '20]

[Lee, '20]: Spin Hurwitz theory

$$S_k(\lambda) := -\frac{B_k}{2k} + \sum_i \lambda_i^{k-1}$$

Objectives for this talk

1. Describe relation to quasimodular forms of (i)-(iv).

2. Describe how to obtain (ii)-(iv) from (i).

3. (if time permits) Introduce $\tilde{J}$.

p-adic analogues of shifted symmetric functions (ii)

Recall

$$Q_k^{(p)}(\lambda) = \beta_k\left(1-\frac{1}{p}\right) + \frac{1}{(k-1)!}\sum_{(2\lambda_i-2i+1,p)=1}\left((\lambda_i - i+\tfrac{1}{2})^{k-1} - (-i+\tfrac{1}{2})^{k-1}\right)$$

Observe

$$Q_k^{(p)} = [z^{k-1}]\,W(z) - \frac{1}{p}\sum_{a=0}^{p-1}\left[\left(z-\frac{2a}{p}\right)^{k-1}\right] W(z)$$

p-adic analogues of shifted symmetric functions (ii)

$$\begin{tikzcd}
\overline{\mathbb{Q}}[W(z_i)] \arrow[r, "\langle\ \rangle_q"] \arrow[d] & \widetilde{\mathcal{J}} \arrow[d] \\
\Lambda^*(p) \arrow[r, "\langle\ \rangle_q"] & \widetilde{M}(p)
\end{tikzcd}$$

"Taylor coefficients around $z_i \in \frac{\mathbb{Z}}{p}$".

Write $\Lambda^*(p) = \overline{\mathbb{Q}}[Q_k(a) : k \geq 0,\ a \in \frac{\mathbb{Z}}{p}]$, $\Lambda^{(p)} = \mathbb{Q}[Q_k^{(t)} : t \mid p]$ $(p \in \mathbb{N})$

Cor For $f \in \Lambda^{(p)}$, we have $\langle f \rangle_q \in \widetilde{M}(\Gamma_0(p^2))$.

t-hook moments (iii)

Recall $$H_k^{(t)} = -\frac{B_k}{2k} t^k + \sum_{\substack{\xi \in Y\lambda \\ t \mid h(\xi)}} h(\xi)^{k-2}$$

Thm (Chen–Möller–Zagier)

$$\frac{1}{(k-2)!} H_k^{(1)}(\lambda) = -\frac{1}{2} [z^{k-1}]\, W_\lambda(z)\, W_\lambda(-z)$$

Write $\mathcal{H}^{(N)} = \mathbb{Q}[H_k^{(t)} \mid k \geq 0 \text{ even}, t \mid N]$

Cor For $f \in \mathcal{H}^{(N)}$ we have $\langle f \rangle_q \in \widetilde{M}(\Gamma_0(N^2))$.

Note For k odd/negative $\langle H_k^{(t)} \rangle_q$ can be understood as quantum/Harmonic Maass forms. What about say, $\langle H_{k_1}^{(t_1)} H_{k_2}^{(t_2)} \rangle_q$?

Moment functions (iv)

Recall $\quad S_k(\lambda) = -\frac{B_k}{2k} + \sum_i \lambda_i^{k-1}.$

Def For $f : \mathcal{P} \to \overline{\mathbb{Q}}$, we define the Möller tranform of f by

$$\mathcal{M}f(\lambda) = \frac{1}{n!} \sum_{\mu \in \mathcal{P}_n} |C_\mu| \, \chi_\lambda(\mu)^2 f(\mu) \qquad (\lambda \in \mathcal{P}_n)$$

size of conj. class given by μ

character of representation of S_n given by λ at permutation of type μ

Lemma $\quad \langle \mathcal{M}f \rangle_q = \langle f \rangle_q$

Prop $\quad \mathcal{M} S_k = H_k$

Non-example $\quad \mathcal{M} S_{k_1} S_{k_2} \neq H_{k_1} H_{k_2}$, in fact

$$\notin \Lambda^*.$$

Moment functions (iv)

Write $\mathcal{S} = \overline{\mathbb{Q}}[S_k,\ k \geq 2 \text{ even}]$, $\operatorname{wt} S_k = k$

Two results on $\mathcal{S}$ (vI) (a) For all $f \in \mathcal{S}_k$ we have $\langle f \rangle_q \in \widetilde{M}_k$.

(b) $\mathcal{S}$ has a natural extension $\mathcal{T}$ obtained by the isomorphism

$$\langle\ \rangle_{\underline{u}} : \overline{\mathbb{Q}}^{\mathcal{P}} \xrightarrow{\ \sim\ } \overline{\mathbb{Q}}[[u_1, u_2, \ldots]]$$

$$f \longmapsto \frac{\sum f(\lambda)\, u_\lambda}{\sum u_\lambda} \qquad (u_\lambda = u_{\lambda_1} u_{\lambda_2} \cdots)$$

That is, $\mathcal{T}$ is a graded algebra s.t.

1. $\mathcal{S} \subseteq \mathcal{T}$
2. $\mathcal{T}$ and $\langle \mathcal{T} \rangle_{\underline{u}}$ are closed under multiplication
3. $f \in \mathcal{T} \to \langle f \rangle_q \in \widetilde{M}$.

Quasi-Jacobi forms

Recall F_n can be expressed in terms of ratios of derivatives of Θ.

Even so,

n-point function corr. to $\mathcal{J}$

$$G_n(u_1, \ldots, u_n, v_1, \ldots, v_n) = \prod_i \left(-\frac{1}{2} \frac{\Theta(u_i + v_i)}{\Theta(u_i)\Theta(v_i)} \right).$$

Note

$$E_2 | \gamma (\tau) = E_2 - \frac{1}{4\pi i} \frac{c}{c\tau + d}$$

$$\Theta' | \gamma (\tau, z) = \Theta'(\tau, z) + \Theta(\tau, z) \frac{cz}{c\tau + d}$$

slash action as on modular forms / Jacobi forms

Quasi-Jacobi forms

$z \mapsto \varphi(\tau, z)$ admits a pole at $z = X^t \binom{\tau}{1} \in \mathbb{R}^n \tau + \mathbb{R}^n$

Def A strictly meromorphic quasi-Jacobi form of weight k and index $M \in M_n(\mathbb{Q})$ is a meromorphic function $\varphi : \mathfrak{H} \times \mathbb{C}^n \to \mathbb{C}$ s.t.

(i) For all $X \in M_{2,n}(\mathbb{R})$, either $\varphi|_M X$ admits a pole at $(\tau, 0)$ for all generic $\tau \in \mathfrak{H}$ or $\tau \mapsto (\varphi|_M X)(\tau, 0) \in \mathrm{Holo}(\mathfrak{H})$

holomorphic at $\mathfrak{H}$ and all cusps.

(ii) There exist meromorphic $\varphi_{i,\underline{j}} : \mathfrak{H} \times \mathbb{C}^n \to \mathbb{C}$ s.t. $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $X = \begin{pmatrix} \underline{\lambda} \\ \underline{\mu} \end{pmatrix} \in M_{2,n}(\mathbb{Q})$

$$(\varphi|_{k,M}\gamma)(\tau, z_1, \dots, z_n) = \sum_{i,\underline{j}} \varphi_{i,\underline{j}}(\tau, z_1, \dots, z_n) \left(\frac{c}{c\tau + d}\right)^{i + j_1 + \dots + j_n} z_1^{j_1} \cdots z_n^{j_n}$$

$$(\varphi|_M X)(\tau, z_1, \dots, z_n) = \sum_{i,\underline{j}} \varphi_{0,\underline{j}}(\tau, z_1, \dots, z_n)(-\lambda_1)^{j_1} \cdots (-\lambda_n)^{j_n}$$

standard slash action as on Jacobi forms

Key property Taylor coeffient of $\varphi|X$ for $X \in M_{2,n}(\mathbb{Q})$ are quasimodular if $\underline{\lambda} = \underline{0}$

Main result (vI)

Given $V: \mathcal{P} \times \mathbb{C}^r \to \overline{\mathbb{Q}}$ s.t.

$$\begin{tikzcd}
\overline{\mathbb{Q}}[V(\cdot, z_i)] \arrow[r] \arrow[d, dashed] & \tilde{\mathcal{J}} \arrow[d, dashed] \\
\mathcal{F}(N) \arrow[r] & \widetilde{M}(N)
\end{tikzcd}$$

algebra of $f: \mathcal{P} \to \overline{\mathbb{Q}}$ with $\langle f \rangle_q \in \widetilde{M}(N)$

"Taylor coefficients at $z_i \in \frac{\mathbb{Z}}{N}$"

Example

$$S_k^{(t)} := -\frac{B_k}{2k} + \sum_{\substack{i \geq 0 \\ t \mid \lambda_i}} \lambda_i^{k-1}$$

For $f \in \overline{\mathbb{Q}}[S_k^{(t)} \mid k \geq 2 \text{ even}, t \mid N]$, $\langle f \rangle_q \in \widetilde{M}(\Gamma_0(N^2))$.

Thank you!

Bonus: Taylor coefficients of quasi-Jacobi forms

Example Given $X \in M_{2,1}(\mathbb{Q})$, there is a group $\Gamma_X \leq SL_2(\mathbb{Z})$ s.t.

$$(\Theta'|X)|_\gamma = \Theta'|X + \Theta|X \frac{cz}{c\tau+d} + \underbrace{\lambda\,\Theta|X - \frac{\lambda}{c\tau+d}\Theta|X}\,,$$

where $X = \begin{pmatrix}\lambda\\ \mu\end{pmatrix}$, $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_X$.

Upshot: Taylor coefficients of $\Theta' + \lambda\Theta$ around $z = \lambda\tau + \mu$ are quasimodular (rather than of Θ').

Def

$$\varphi \| X := \sum_{\underline{j}} \varphi_{0,\underline{j}} | X \;\lambda_1^{j_1} \cdots \lambda_n^{j_n}$$

(φ quasi-Jacobi of integral index)

Taylor coefficients of $\varphi \| X$ (rather than of $\varphi | X$) are quasimodular.

Bonus: poles of quasi-Jacobi forms φ

Recall $z = X^t\binom{\tau}{1}$ is a pole of φ for some $\tau \in \mathfrak{h}$, then for all generic $\tau \in \mathfrak{h}$.

<u>Thm</u> (vI) All such poles of a given φ lie in a finite union of rational hyperplanes

$$s_1 z_1 + \dots + s_n z_n \in p\tau + q \qquad (s_i \in \mathbb{Z},\ p, q \in \mathbb{Q}/\mathbb{Z})$$

Bonus: explicit description of $\mathcal{T}$

Denote $r_m(\lambda) = \#\{i \mid \lambda_i = m\}$ $\quad (\lambda \in \mathcal{P}, m \geq 1)$

Then $\mathcal{T}$ is generated by $\quad (k+\ell \geq 2$ even$)$

$$T_{k,\ell} = -\frac{B_{k+\ell}}{2(k+\ell)}(\delta_{k,0} + \delta_{\ell,1}) + \sum_{m\geq 1} m^k f_\ell(r_m(\lambda))$$

unique polynomial s.t.

- $f_\ell(n) - f_\ell(n-1) = n^{\ell-1} \quad (n \in \mathbb{Z}_{\geq 1})$
- $f_\ell(0) = 0$