On quasimodular forms associated to projective representations of symmetric groups

International seminar on automorphic forms

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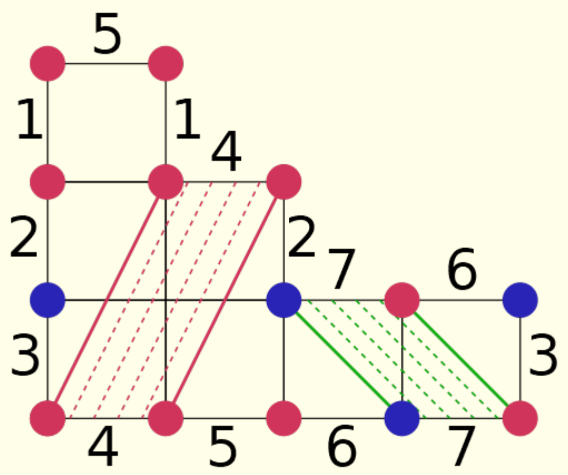
Overview

① Determine invariants in some moduli space



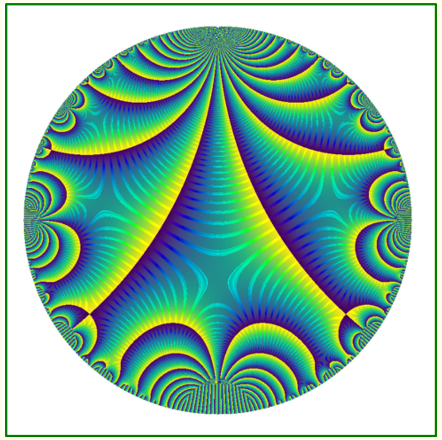
monodromy representation

② Problem in asymptotic representation theory

$$|C_\mu| \frac{\chi_\lambda(\mu)}{\chi_\lambda(e)}$$

generating series

③ (Quasi) modular forms



growth polynomials

④ Asymptotic statements about these invariants

Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials

Alex Eskin[1], Andrei Okounkov[2]

The theta characteristic of a branched covering

Alex Eskin[a], Andrei Okounkov[b,*], Rahul Pandharipande[b]

QUASIMODULARITY AND LARGE GENUS LIMITS OF SIEGEL-VEECH CONSTANTS

DAWEI CHEN, MARTIN MÖLLER, AND DON ZAGIER

Hurwitz theory of elliptic orbifolds, I

PHILIP ENGEL

Masur–Veech volumes and intersection theory on moduli spaces of Abelian differentials

Dawei Chen[1] · Martin Möller[2] · Adrien Sauvaget[3] · Don Zagier[4]

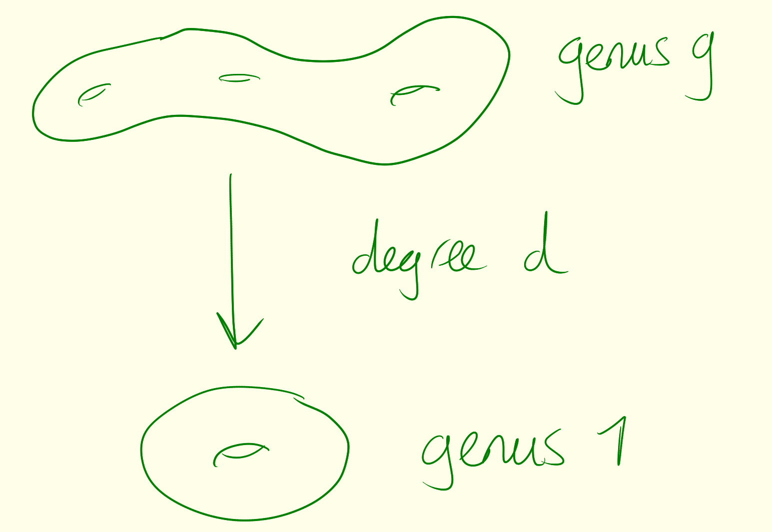
Hurwitz numbers / Mirror symmetry of dim. 1 (Dijkgraaf)

Say $h = (\alpha, \beta, \gamma_1, \dots, \gamma_{2g-2}) \in \mathfrak{S}_d^{2g}$ is a Hurwitz tuple if

$\leftarrow$ symmetric group

- $[\alpha, \beta]\, \gamma_1 \cdots \gamma_{2g-2} = 1$;
- γ_i are transpositions;
- $\langle \alpha, \beta, \gamma_1, \dots, \gamma_{2g-2} \rangle$ acts transitively on $\{1, \dots, d\}$.

don't want solutions from $\mathfrak{S}_n \times \mathfrak{S}_m \subseteq \mathfrak{S}_{n+m}$



$$h_{g,d} := \frac{1}{d!} \left| \{ h \in \mathfrak{S}_d^{2g} \text{ Hurwitz tuple} \} \right|$$

$$H_g(q) := \sum_{d=1}^{\infty} h_{g,d}\, q^d$$

Ex $H_2(q) = 2q^2 + 16q^3 + 60q^4 + 160q^5 + 360q^6 + 672q^7 + 1240q^8 + \dots$

$= \frac{1}{3}(DG_4 - D^2 G_2)$, where

- $D := q \frac{\partial}{\partial q}$
- $G_k := -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n$
- $\widetilde{M} := \mathbb{Q}[G_2, G_4, G_6]$

Thm (Dijkgraaf, Kaneko-Zagier)

$H_g \in \widetilde{M}_{\leq 6g-6}$ for $g \geq 2$.

Quasimodular forms associated to symmetric group representations

Let $\rho_\lambda : \mathfrak{S}_d \to GL_n(\mathbb{C})$ be irreducible. Given a partition μ of d, consider $\sum_{\sigma\in C_\mu} \sigma$ in the center of $\mathbb{C}[\mathfrak{S}_d]$. Then, by Schur's lemma, $\rho_\lambda\left(\sum_{\sigma\in C_\mu}\sigma\right)$ acts by a constant, called the central character $f_\mu(\lambda)$. We extend this character to all partitions λ,μ by

$$f_\mu(\lambda) = \binom{|\lambda|-|\mu|+r_1(\mu)}{r_1(\mu)} f_{\mu,1,\dots,1}(\lambda)$$

with $|\lambda|, |\mu|$ size of λ,μ, $r_1(\mu)$ multiplicity of 1 in μ.

Rk $f_\mu(\lambda) = |C_\mu| \dfrac{\chi_\lambda(\mu)}{\chi_\lambda(e)}$ if $1\notin\mu$.

Given $f : P \to \mathbb{C}$, we define the q-bracket of f by

$$\langle f\rangle_q := \frac{\sum_{\lambda\in P} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in P} q^{|\lambda|}} \in \mathbb{C}[[q]]$$

Ex $f_1(\lambda) = \binom{|\lambda|-1+1}{1} f_{1,\dots,1}(\lambda) = |\lambda|$ and

$$\langle f_1\rangle_q = \frac{\sum_{\lambda\in P} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda\in P} q^{|\lambda|}} = q\frac{\partial}{\partial q}\log \overbrace{\sum_{\lambda\in P} q^{|\lambda|}}^{\prod_{n\geq 1}(1-q^n)^{-1}}$$

$$= \sum_{n\geq 1}\sigma(n)\, q^n = \frac{1}{24} + G_2$$

Quasimodular forms associated to symmetric group representations (II)

Ex By the Frobenius formula

$$H_2 = \langle f_2^{\,2}\rangle_q - \langle f_2\rangle^2$$

$$H_3 = \langle f_2^{\,4}\rangle_q - 4\langle f_2^{\,3}\rangle_q\langle f_2\rangle_q - 3\langle f_2^{\,2}\rangle_q^2 + 12\langle f_2^{\,2}\rangle\langle f_2\rangle_q^2 - 6\langle f_2\rangle_q^4$$

$$H_g = \sum_{\nu\in\mathcal{P}(2g-2)} (-1)^{\ell(\nu)-1}(\ell(\nu)-1)!\,|C_\nu|\prod_i \langle f_2^{\,\nu_i}\rangle_q$$

length of ν — size of conjugacy class associated to ν

Thm (Kerov) For any partition μ, one has $f_\mu \in \Lambda^*$.

Here, Λ^* is the algebra of shifted symmetric functions, freely generated by $Q_k : \mathcal{P} \to \mathbb{Q}$

$$Q_0(\lambda)=1,\quad Q_1(\lambda)=0,\quad Q_k(\lambda) = -\left(1-\frac{1}{2^{k-1}}\right)\frac{B_k}{k} + \sum_{i=1}^{\infty}\left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1}\right) \quad (k\geq 2)$$

Ex $f_1 = Q_2 + \frac{1}{24}$, $f_2 = \frac{1}{2}Q_3$, $f_3 = \frac{1}{3}Q_4 - \frac{1}{2}Q_2^2 + \frac{3}{8}Q_2 + \frac{9}{640}$, ...

Thm (Bloch–Okounkov) For all $f \in \Lambda^*_k$ one has $\langle f\rangle_q \in \widetilde{M}_k$

Ex $H_2 = \langle f_2^2\rangle_q - \langle f_2\rangle_q^2$ with $\langle f_2\rangle_q = 0$ and $\langle f_2^2\rangle_q = \frac{1}{3}(DG_4 - DG_2^2)$.

Quasimodular forms associated to symmetric group representations (III)

Thm (Kerov) For any partitition μ, one has $f_\mu \in \Lambda^*$

Let $h_\ell = -\frac{1}{\ell^2}[u^{\ell+1}]\, P(u)^\ell$, with $P(u) = \exp\left(-\sum_{s\geq 2} u^s Q_s\right)$

Thm (Eskin-Okounkov) $f_\ell \equiv h_\ell \mod \Lambda^*_{\leq \ell-1}$

Thm (Bloch-Okounkov) For all $f \in \Lambda^*_k$ one has $\langle f\rangle_q \in \widetilde{M}_k$, more precisely,

$$F_n(z_1,\dots,z_n) := \sum_{k_1,\dots,k_n\geq 2} \langle Q_{k_1}\cdots Q_{k_n}\rangle_q \frac{z_1^{k_1-1}}{(k_1-1)!}\cdots\frac{z_n^{k_n-1}}{(k_n-1)!}$$ satisfies

derivative in elliptic variable

$$F_n(z_1,\dots,z_n) = \sum_{\sigma\in S_n} \frac{\det\left[\frac{\theta^{(j-i+1)}(x_{\sigma(1)}+\dots+x_{\sigma(n-j)})}{(j-i+1)!}\right]^n}{\theta(x_{\sigma(1)})\,\theta(x_{\sigma(1)}+x_{\sigma(2)})\cdots\theta(x_{\sigma(1)}+\dots x_{\sigma(n)})};\quad \theta(z) = \sum_{n\in\mathbb{Z}}(-1)^n e^{(n+\frac12)z} q^{\frac{(n+\frac12)^2}{2}}.$$

Thm (Zagier)

$$\frac{\partial}{\partial G_2}\langle f\rangle_q = \langle \mathcal{D} f\rangle_q \text{ with } \mathcal{D} = \sum_{a,b\geq 1}\left(-\binom{a+b}{a} Q_{a+b} + Q_a Q_b\right)\frac{\partial}{\partial Q_{a+1}\partial Q_{b+1}} + \sum_{a\geq 0} Q_a \frac{\partial}{\partial Q_{a+2}}$$

Growth polynomials of quasimodular forms

quasimodular forms

Two questions about asymptotics of $F \in \widetilde{M}$, $F: \mathfrak{h} \to \mathbb{C}$, $\tau \mapsto F(\tau)$ with $q = e^{2\pi i \tau}$.

① How does F behave around a cusp, e.g., $F\left(-\frac{h}{2\pi i}\right) \sim\ ?$ as $h \to 0$

② How do Fourier coefficients of F grow, i.e. $\sum_{n=1}^{N} a_n(F) \sim\ ?$ as $N \to \infty$

Prop (Chen–Möller–Zagier) For $F \in \widetilde{M}_{2k}$

- $F\left(-\frac{h}{2\pi i}\right) \sim A h^{-p} + O(h^{1-p}) \Longrightarrow \sum_{n=1}^{N} a_n(F) \sim A \frac{N^p}{p!} + O(N^{p-1} \log N)$

- $F\left(-\frac{h}{2\pi i}\right) = \frac{1}{h^k} \mathrm{Ev}[F]\left(\frac{(2\pi i)^2}{h}\right) + o(1)$ as $h \to 0$,

where $\mathrm{Ev}: \widetilde{M} \to \mathbb{Q}[X]$ is the algebra hom given by

$$E_2 \mapsto X + 12 \qquad E_4 \mapsto X^2 \qquad E_6 \mapsto X^3$$

with $E_k = \left(-\frac{B_k}{2k}\right)^{-1} G_k = 1 - \frac{2k}{B_k} \sum_n \sigma_{k-1}(n) q^n$

Growth polynomials of quasimodular forms (II)

Ex $\langle f_1\rangle_q = G_2 + \frac{1}{24}$; $\mathrm{Ev}\langle f_1\rangle_q = -\frac{1}{24}X - \frac{1}{2} \Rightarrow \sum_{n=1}^{N}\sigma(n) \sim -\frac{1}{24}(-4\pi^2)\frac{N^2}{2} + O(N\log N)$

$\langle f_2^2\rangle_q = \frac{1}{3}(DG_4 - D^2G_2)$; $\mathrm{Ev}\langle f_2\rangle_q^2 = \frac{X^2}{180} + \frac{X}{12} + \frac{1}{3}$ $\qquad = \frac{\zeta(2)}{2}N^2 + O(N\log N)$

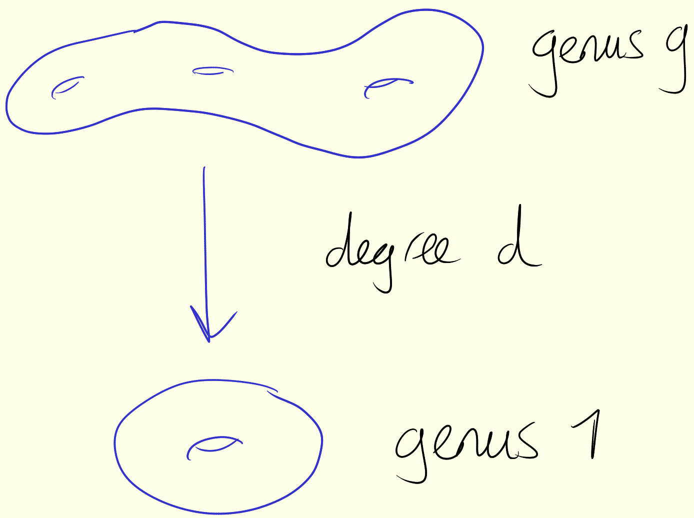
$$\sum_{d=1}^{N} h_{2,d} \sim \frac{1}{180}(16\pi^4)\frac{N^5}{5!} + O(N^4\log N) = \frac{1}{15}\zeta(4)N^5 + O(N^4\log N) \quad (N\to\infty)$$

Prop (Chen–Möller–Sauvaget–Zagier)
For $f \in \Lambda^*_{2k}$ $\quad \mathrm{Ev}(\langle f\rangle_q) = X^k\left(e^{-\frac{1}{2}\mathcal{D}/X} f\right)(\emptyset)$, where we recall

$$\begin{cases} \dfrac{\partial}{\partial G_2}\langle f\rangle_q = \langle \mathcal{D}f\rangle_q \quad \text{with} \\[2ex] \mathcal{D} = \displaystyle\sum_{a,b\geq 1}\left(-\binom{a+b}{a}Q_{a+b} + Q_aQ_b\right)\frac{\partial^2}{\partial Q_{a+1}\partial Q_{b+1}} + \sum_{a\geq 0} Q_a\frac{\partial}{\partial Q_{a+2}} \end{cases}$$

Summary

1 Determine invariants $h_{g,d}$



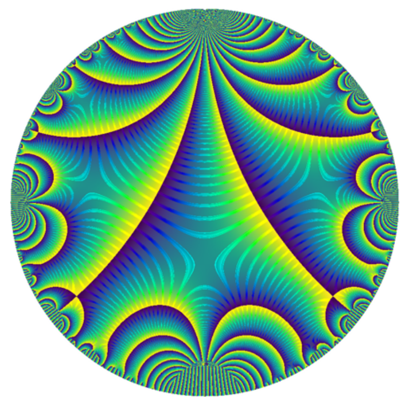
monodromy representation

2 Problem in asymptotic representation theory

$$f_\mu(\lambda) = |C_\mu| \frac{\chi_\lambda(\mu)}{\chi_\lambda(e)}$$

q-bracket $\langle\ \rangle_q$

3 (Quasi) modular forms



growth polynomials

4 Asymptotic statements about $h_{g,d}$

Weighted Hurwitz tuples

Recall $h = (\alpha, \beta, \gamma_1, \dots, \gamma_{2g-2}) \in \mathfrak{S}_d^{2g}$ is a Hurwitz tuple for $(\mu^{(1)}, \dots, \mu^{(2g-2)})$ if

- $[\alpha, \beta]\, \gamma_1 \cdots \gamma_{2g-2} = 1$;
- γ_i are ~~transpositions~~ of cycle type $\mu^{(i)}$
- $\langle \alpha, \beta, \gamma_1, \dots, \gamma_{2g-2} \rangle$ acts transitively on $\{1, \dots, d\}$.

$$h^{\mu}_{g,d}(w) := \sum_{\substack{h \in \mathfrak{S}_d^{2g} \\ \text{Hurwitz tuple for } \mu}} w(h)$$ for some weight function w.

Ex $w(h) = (-1)^{s(h)} V(\alpha)$ with $s(h) \in \mathbb{Z}/2\mathbb{Z}$ the theta characteristic. (defined later)

and $V : \mathfrak{S}_d \to \mathbb{C}$ a class function

Variation: projective representations

Let $\rho : \mathfrak{S}_d \to PGL_n(\mathbb{C})$ be an irreducible projective representation, not restricting to an ordinary representation. Such representations are parametrized by strict partitions λ, together with a sign $\pm$. Equivalently, one studies ordinary irreducible representations of

$$\mathrm{Se}_d := \mathfrak{S}_d \ltimes \mathrm{Cl}_d \,, \quad \mathrm{Cl}_d := \{\xi_1, \dots, \xi_d, \varepsilon \mid \varepsilon^2 = 1, \ \xi_i^2 = \varepsilon, \ \varepsilon \xi_i = \xi_i \varepsilon, \ \xi_i \xi_j = \xi_j \xi_i \varepsilon \ \ \forall i \neq j\}$$

Similarly, one constructs central characters $f_\mu(\lambda)$ for λ strict and μ odd.

Thm (Ivanov) $f_\mu \in \mathcal{S}^{odd}$, algebra of odd symmetric functions, generated by $p_k = -\frac{1}{2}\zeta(-k) + \sum_i \lambda_i^k$ (k odd)

Thm (Eskin–Okounkov) For all $f \in \mathcal{S}^{odd}$ one has $\langle f \rangle^{spin} \in \widetilde{M}$, where

$$\langle f \rangle^{spin} = \frac{\sum_{\lambda \in SP} (-1)^{\ell(\lambda)} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in SP} (-1)^{\ell(\lambda)} q^{|\lambda|}}$$

set of strict partitions

Rk Also other results we saw previously have analogous versions.

Spin parity

Recall $Se_d := \mathfrak{S}_d \ltimes Cl_d$, $\quad Cl_d := \{\xi_1,\dots,\xi_d,\varepsilon \mid \varepsilon^2=1,\ \xi_i^2=\varepsilon,\ \varepsilon\xi_i=\xi_i\varepsilon,\ \xi_i\xi_j=\xi_j\xi_i\varepsilon\ \forall i\neq j\}$

Assign $\mathrm{wt}\,\xi_i=1$, $\mathrm{wt}\,\sigma=\mathrm{wt}\,\varepsilon=0$ for $\sigma\in\mathfrak{S}_d$

Then

$$\begin{tikzcd}[column sep=small]
 & Se_d \arrow[dl, dash] \arrow[dr, dash] & \\
Se_d/(\varepsilon-1) = B_d \arrow[dr, dash] & & Se_d^0 \arrow[dl, dash] \\
 & B_d^0 \arrow[r, two heads] & \mathfrak{S}_d
\end{tikzcd}$$

"set $\varepsilon=1$"

Se_d^0 ← elements of even weight

number of lifts of h to a Hurwitz tuple for B_d, B_d^0, Se_d, Se_d^0 resp.

Def For a Hurwitz tuple h for $\mathfrak{S}_d^{2g}$ we let

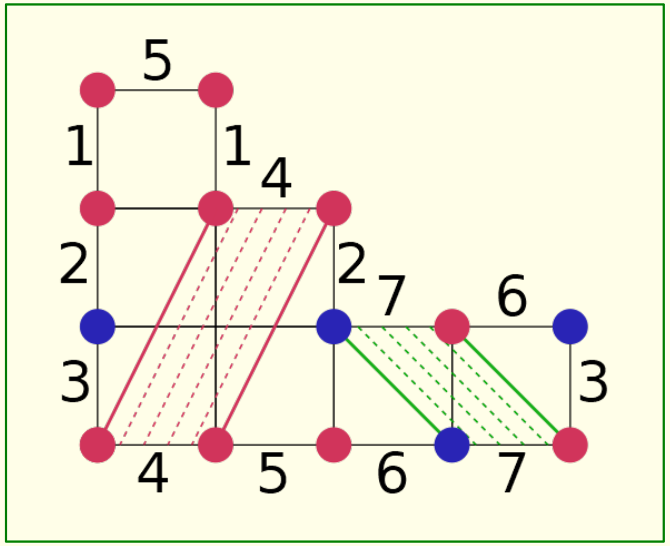
$$\frac{1}{d!}(-1)^{s(h)} := 2^{1-g}\left(\frac{|Hur_{B_d}(h)|}{|B_d|} - \frac{|Hur_{B_d^0}(h)|}{|B_d^0|} - \frac{|Hur_{Se_d}(h)|}{|Se_d|} + \frac{|Hur_{Se_d^0}(h)|}{|Se_d^0|}\right)$$

Prop (Sauvaget-vI)

$$\frac{1}{d!}\sum_{\substack{h\in\mathfrak{S}_d^{2g}\\ \text{Hurwitz tuple}}}(-1)^{s(h)}V(\alpha) = 2^{1-g}\sum_{\substack{\gamma\in\Pi(d)\\ \gamma=(A_1,\dots,A_r)}}(-1)^{r-1}(r-1)!\left\langle\prod_{i\in A_1} f_{\mu^{(i)}}\cdot V\right\rangle^{spin}\prod_{j=2}^{r}\left\langle\prod_{i\in A_j} f_{\mu^{(i)}}\right\rangle^{spin}$$

Corollary

① Determine spin Siegel-Veech constants in moduli space of curves with associated Abelian differential



monodromy representation

② Problem in asymptotic representation theory

$$Se_d$$

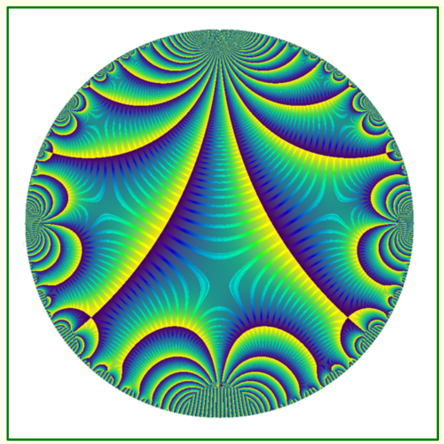
$$B_d \qquad Fu(\lambda) \qquad Se^0_d$$

$$B^0_d$$

spin q-bracket

$\langle\ \rangle_{spin}$

③ (Quasi) modular forms



growth polynomials

④ Asymptotic statements about these invariants

Thank you!