Zeros of quasimodular forms & the sum formula for multiple zeta values

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I Modular forms: zeros & critical points

(jt. with Berend Ringeling)

$\{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$

Def A modular form of weight k is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ s.t.

- $(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$ $\quad \left(\tau \in \mathfrak{h},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\right)$
- $f(\tau) = \sum_{n\geq 0} a_n q^n$ for some $a_n \in \mathbb{Q}$ $\quad (q = e^{2\pi i \tau})$

Ex $E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{m,r\geq 1} m^{k-1} q^{mr} \overset{k>2}{=} \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)=1}} \frac{1}{(m\tau+n)^k}$

is modular for $k>2$.

Bernoulli number

Rk In fact, $M = \mathbb{Q}[E_4, E_6]$

all modular forms

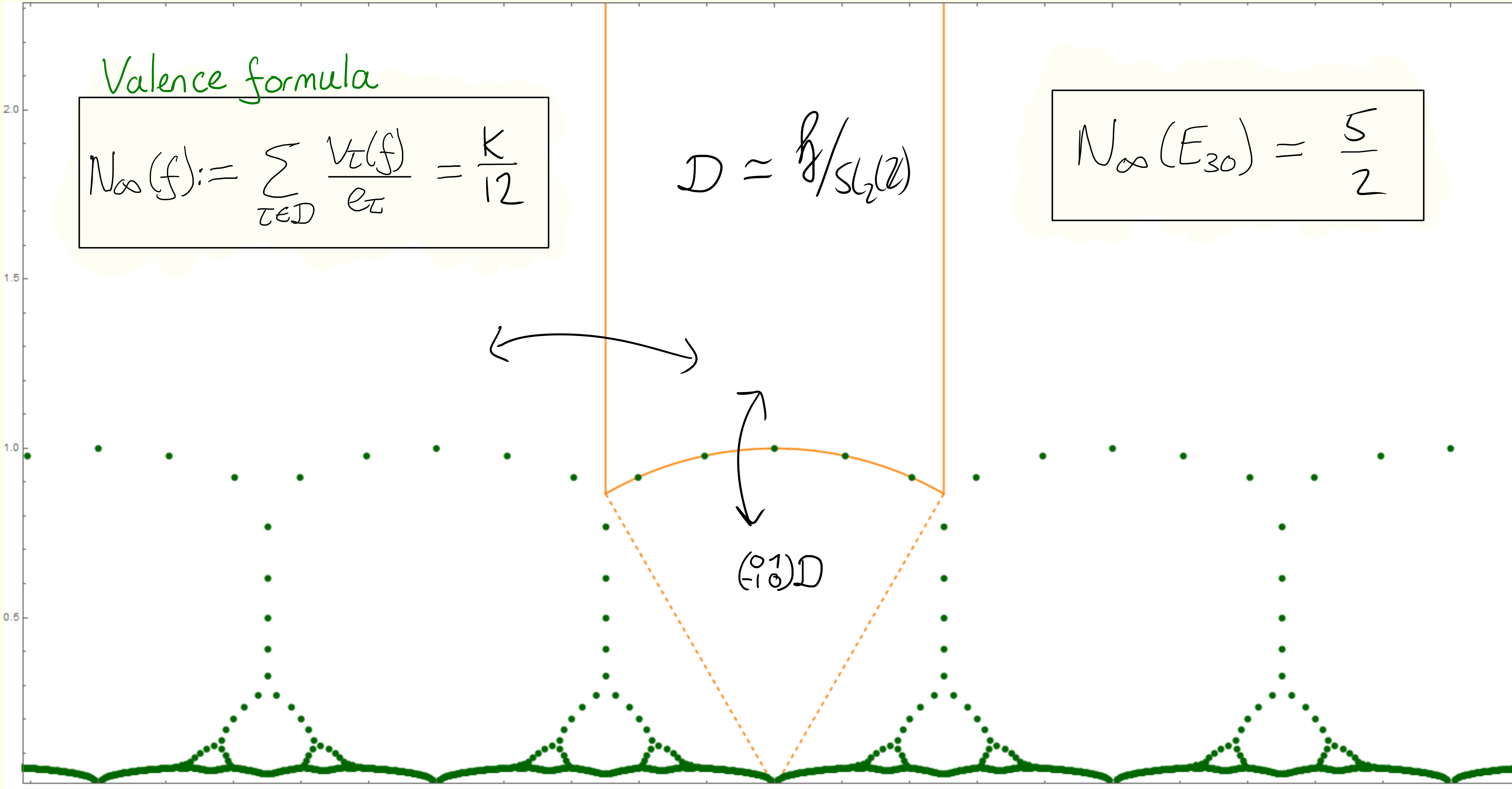
Ex Zeros of E_{30}.

Valence formula

$$\boxed{N_\infty(f) := \sum_{\tau\in\mathcal{D}} \frac{v_\tau(f)}{e_\tau} = \frac{k}{12}}$$

$$\mathcal{D} \simeq \mathfrak{h}/SL_2(\mathbb{Z})$$

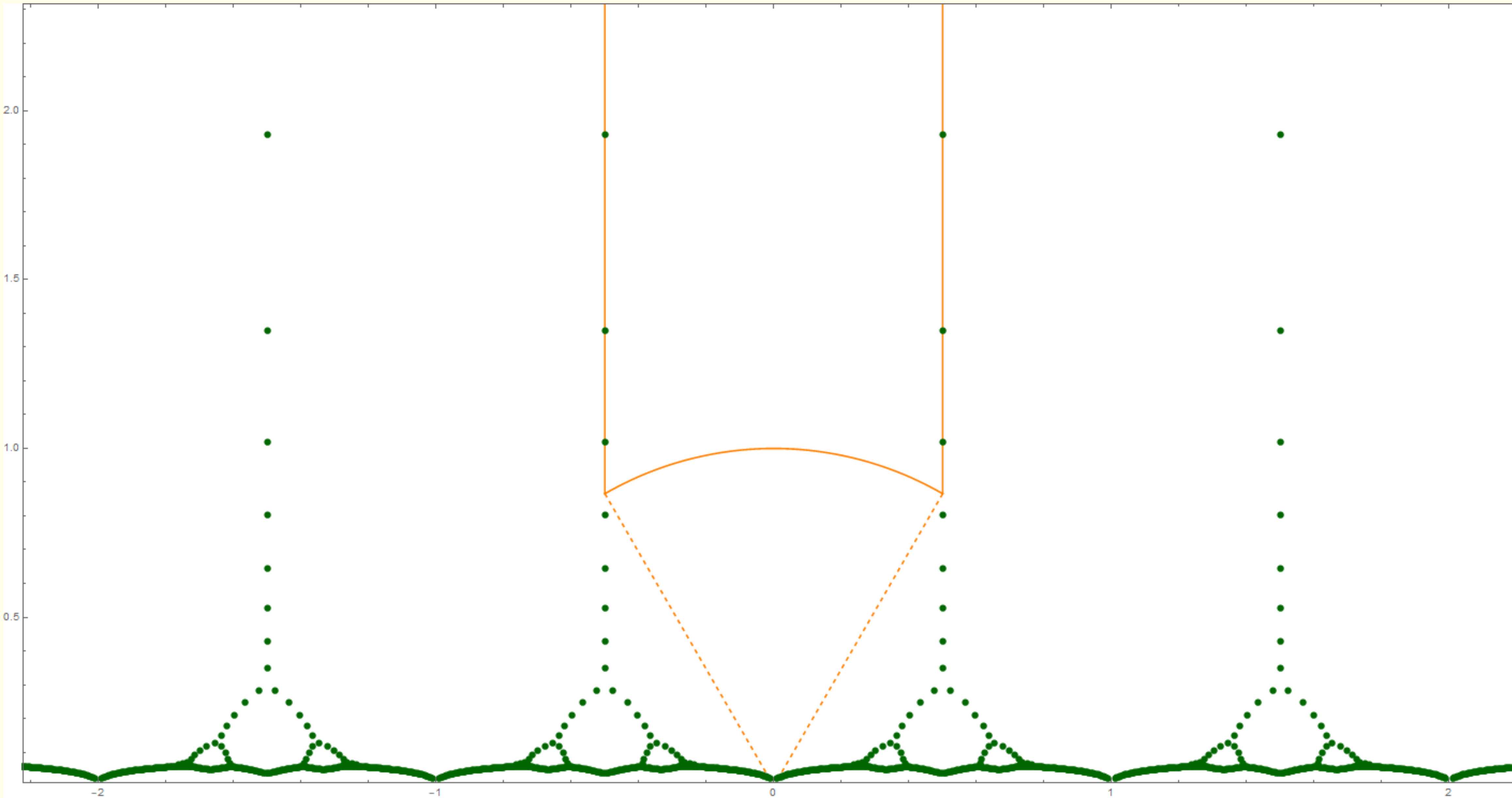
$$\boxed{N_\infty(E_{30}) = \frac{5}{2}}$$



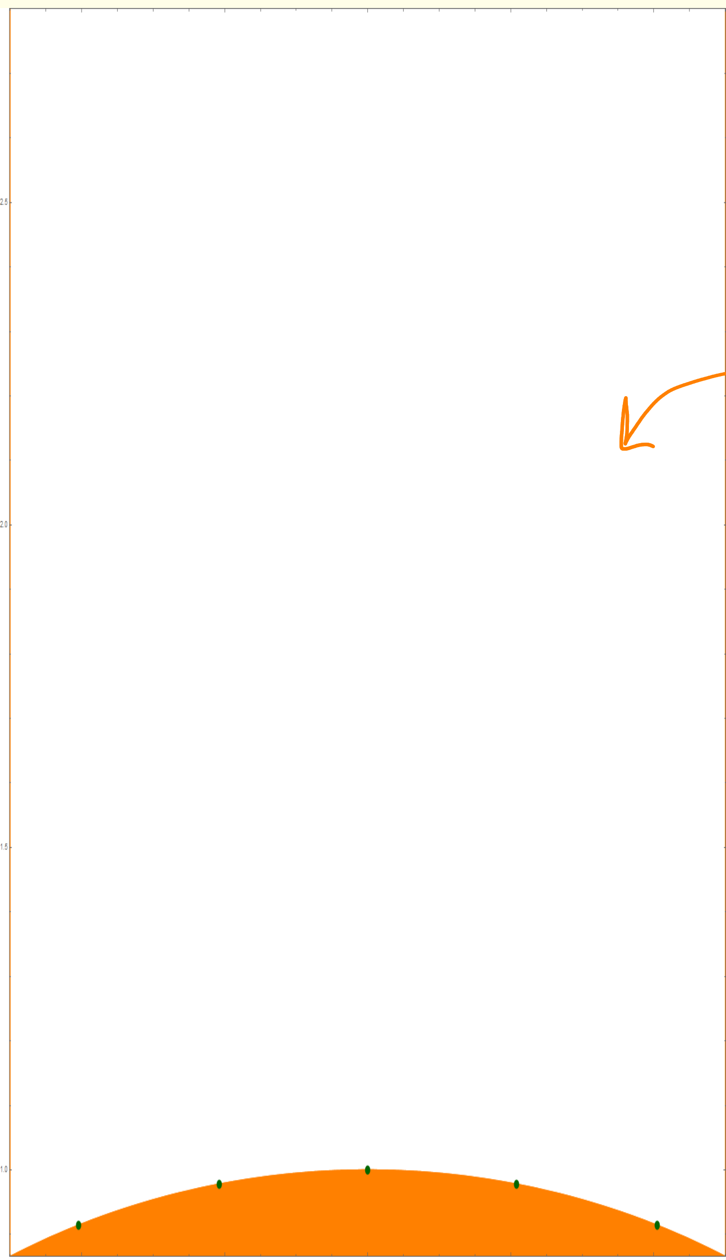
$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\mathcal{D}$$

Ex Critical points of E_{30} (zeros of E_{30}').

first derivative



$SL_2(\mathbb{Z})\tau_0$, for all $\tau_0 \in \mathbb{H}$ with $E_{30}(\tau_0)=0$ (left) or $E_{30}'(\tau_0)=0$ (right)

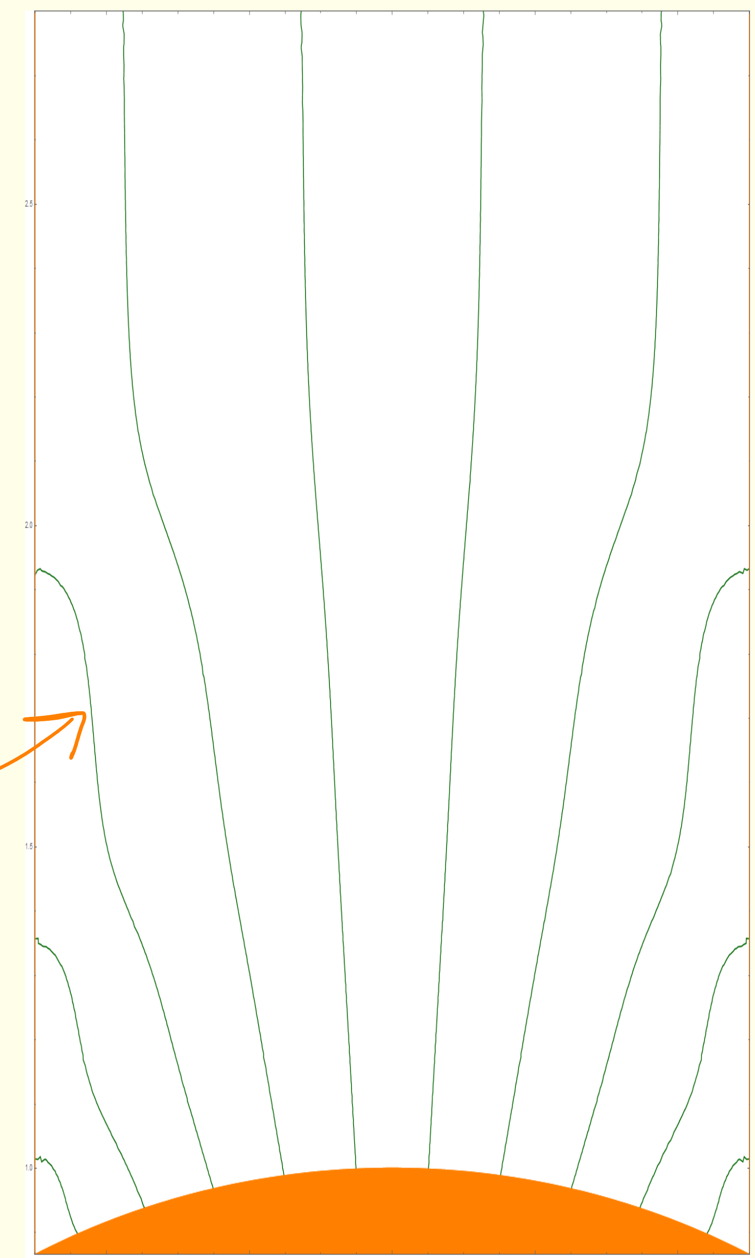


$$(c\tau+d)^{-30} E_{30}\left(\frac{a\tau+b}{c\tau+d}\right) = E_{30}(\tau)$$

$$E_{30}(\tau)=0 \Leftrightarrow E_{30}\left(\frac{a\tau+b}{c\tau+d}\right)=0$$

$$(c\tau+d)^{-32} E_{30}'\left(\frac{a\tau+b}{c\tau+d}\right) = E_{30}'(\tau) + \frac{30}{2\pi i}\,\frac{c}{c\tau+d}\,E_{30}(\tau)$$

$$f'(\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} f(\tau)$$



Thm (Ringeling-vI) Let $f = f_0 + f_1 E_2$ with f_0, f_1 modular and without common zeros (e.g. $f = E_{30}'$)

(i) For all $\lambda = -\frac{d}{c} \in \mathbb{Q}$, $(c,d)=1$

$$N_\lambda(f) := \sum_{\tau \in \gamma \mathcal{D}} \frac{v_\tau(f)}{e_\tau} \qquad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})\right)$$

is well-defined;

(ii) There are $M_1(f), M_2(f), M_3(f) \in \mathbb{Q}$ s.t.

$$N_\lambda(f) = \begin{cases} M_1(f) & 1 < \lambda \leq \infty \\ M_2(f) & \frac{1}{2} < \lambda < 1 \\ M_3(f) & 0 \leq \lambda < \frac{1}{2} \end{cases}$$

weight of f

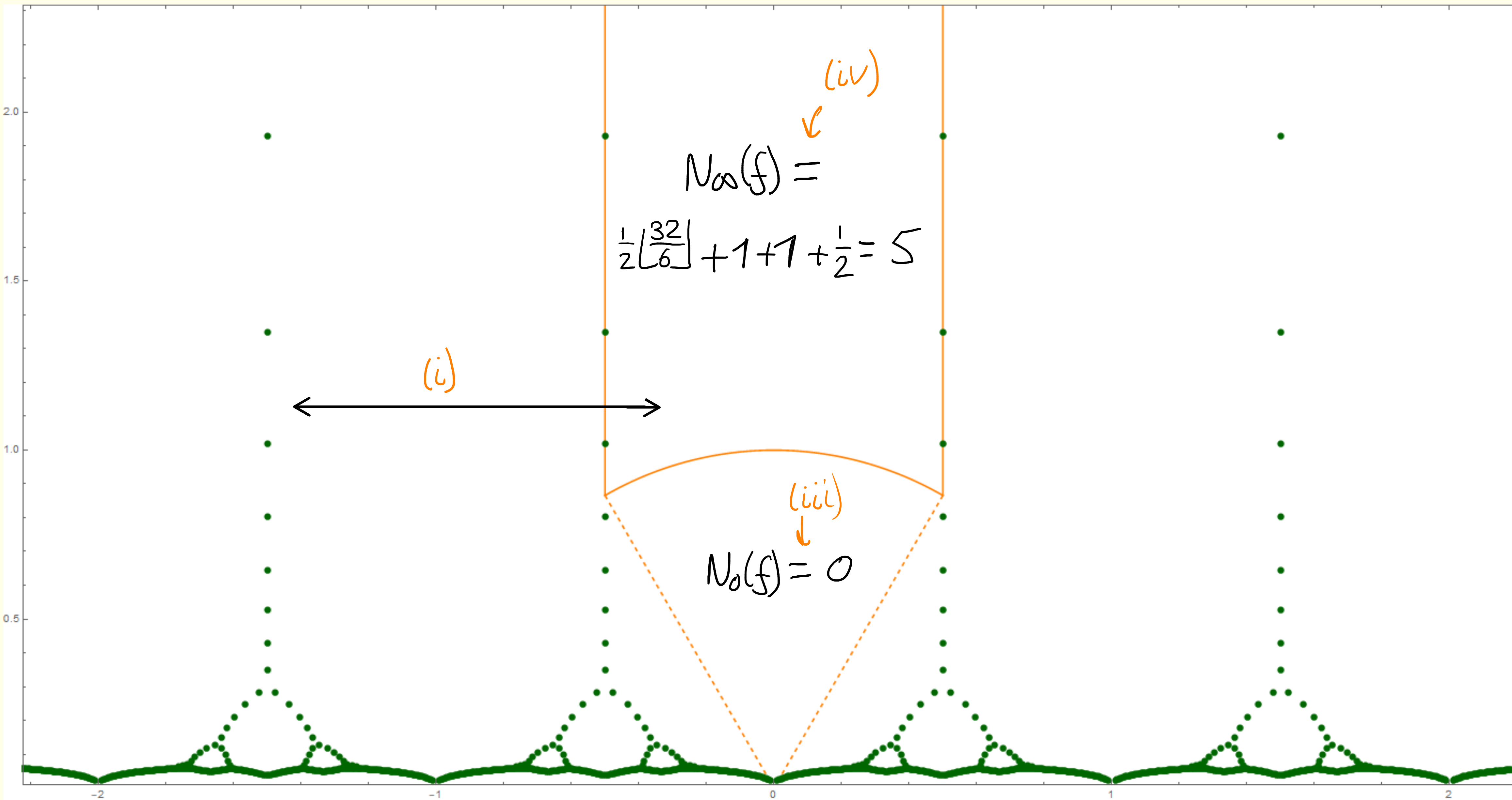
(iii) $M_1(f) + M_2(f) = \lfloor \frac{k}{6} \rfloor$ and $M_3(f) - M_2(f) \in \{0,1\}$

(iv) $M_1(f) = \frac{1}{2}\lfloor \frac{k}{6} \rfloor + \sum_{f_1(e^{i\theta})=0} \text{"}\mathrm{sgn}(f(e^{i\theta}))\text{"}.$

$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$

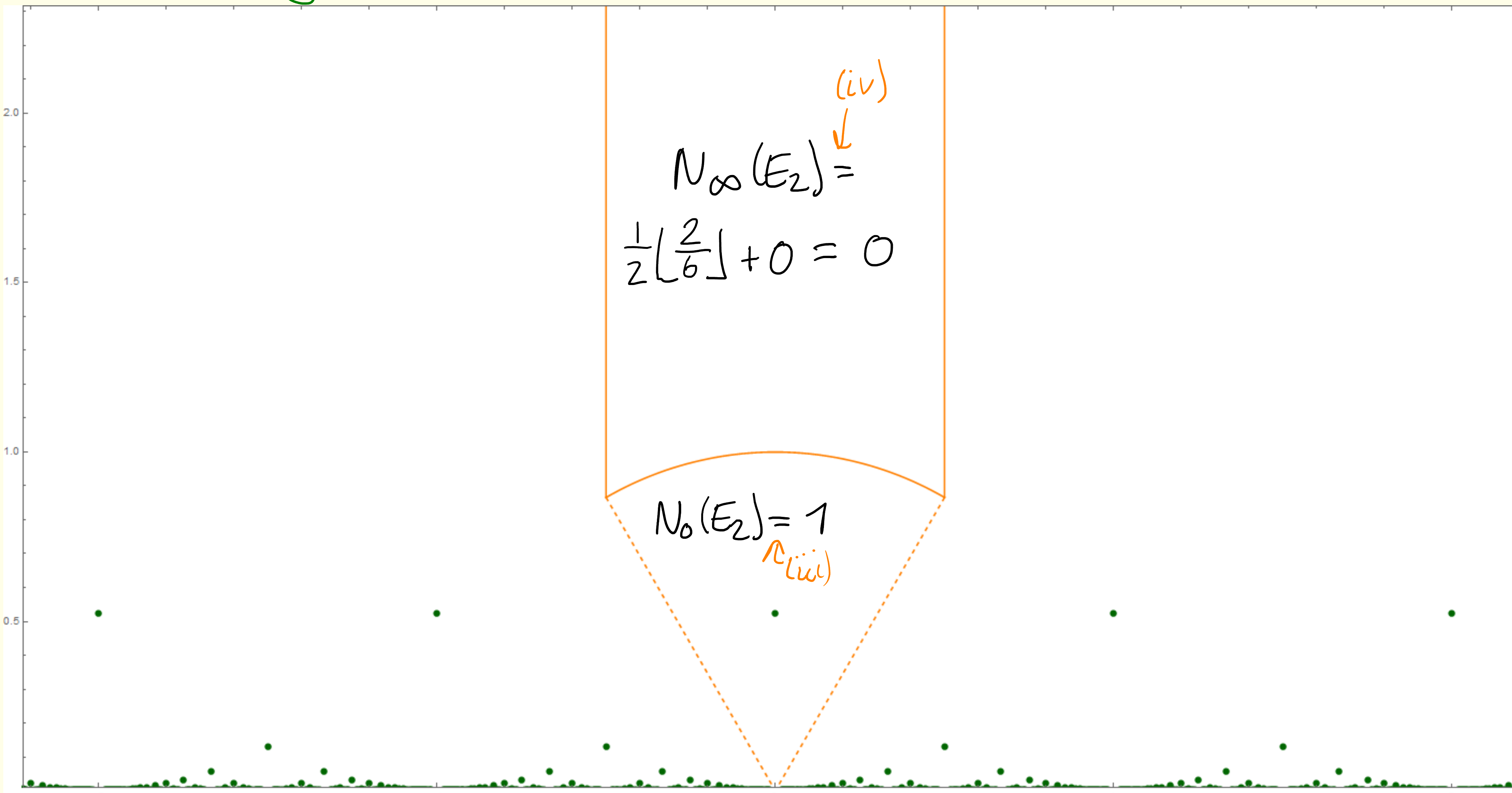
Ex Critical points of E_{30} (zeros of E_{30}').

[Gun-Oesterlé, '20]



Ex Zeros of E_2

[Wood-Young, '13]
[Imamoglu-Jermann-Tóth, '13]



II From functions on partitions to multiple zeta values

(jt. with Henrik Bachmann)

$\mathcal{P}$: set of partitions

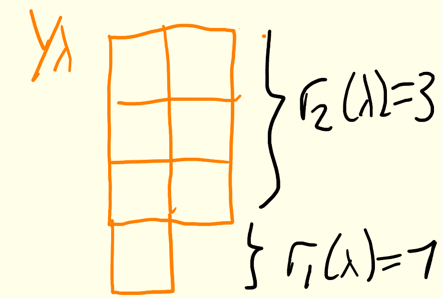
Ex $5 \quad 4+1 \quad 3+2 \quad 3+1+1 \quad 2+2+1 \quad 2+1+1+1+1 \quad 1+1+1+1+1$

Given $f : \mathcal{P} \to \mathbb{Q}$, we define the q-bracket of f by

size $|\lambda| = \sum_i \lambda_i$

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum q^{|\lambda|}} \in \mathbb{Q}[[q]].$$

$\lambda = 2+2+2+1$



Ex $S_2(\lambda) = \sum_m m\, r_m(\lambda) = |\lambda|$

Euler

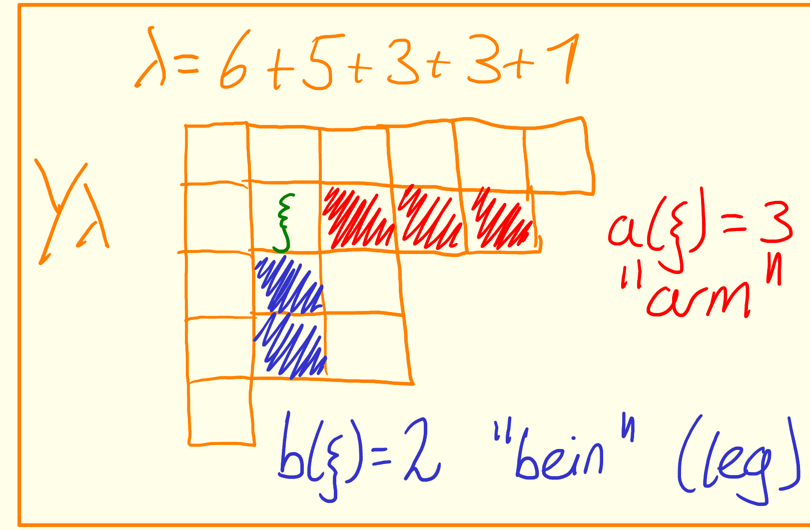
$$\langle S_2 \rangle_q = q\frac{\partial}{\partial q} \log\Big(\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}\Big) = q\frac{\partial}{\partial q} \log \prod_{n\geq 1} (1-q^n)^{-1} = \frac{1-E_2}{24}.$$

Motivation Counting ramified covering of a torus / Gromov-Witten invariants of elliptic curves / Siegel-Veech constants / ... [Dijkgraaf, Kaneko, Zagier, Bloch, Okounkov, Pandharipande, ...]

Today New perspective on relations between MZV's.

Ex Arm- and leg moments $\rightsquigarrow$ sum formula for MZV

$$A_{k,l}(\lambda) := \frac{1}{(k-1)!(l-1)!} \sum_{\xi \in Y_\lambda} \left(a(\xi)+\tfrac{1}{2}\right)^{k-1} \left(b(\xi)+\tfrac{1}{2}\right)^{l-1}$$



By [Zagier, '16], for $k+l$ even

$$\langle A_{k,l} \rangle_q \in \mathbb{Q}[E_2, E_4, E_6].$$

space of quasimodular forms

Rk For all quasimodular f and $K \in \mathbb{Z}$

$$\lim_{q \to 1} (1-q)^K f(\tau) \in \mathbb{Q}[\zeta(2)] \cup \{\infty\}.$$

$(q \to 1 \Leftrightarrow \tau \downarrow 0)$

single zeta value

$\pi^2/6$

Def The space $\mathcal{Z}$ of <u>multiple zeta values</u> is the $\mathbb{Q}$-vector space generated by

$$\zeta(k_1, \dots, k_n) := \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}} \qquad (\in \mathbb{R})$$

for $k_1 \geq 2$, $k_i \geq 1$.

One computes $\lim_{q\to 1}(1-q)^{k+l}\langle A_{k,l}\rangle_q = \zeta(k+l)$.

Also, $$A_{k,l} \sim \frac{1}{k!}\sum_{m_1>\dots>m_l>0}\left(m_1^{k+2} - (m_1-m_l)^{k+2}\right) r_{m_1}(\lambda)\dots r_{m_l}(\lambda).$$

This implies $$\lim_{q\to 1}(1-q)^{k+l}\langle A_{k,l}\rangle_q = \sum_{\substack{k_1+\dots+k_l=k+l\\ k_i\geq 1,\ k_1\geq 2}}\zeta(k_1,\dots,k_l).$$

Thm (sum formula, Granville, Zagier, '98) For all $k,l\geq 1$
$$\sum_{\substack{k_1+\dots+k_l=k+l\\ k_i\geq 1,\ k_1\geq 2}}\zeta(k_1,\dots,k_l) = \zeta(k+l).$$

Ex $\zeta(2,\underbrace{1,\dots,1}_{l-1}) = \zeta(l+1)$.

Ex Shifted symmetric functions $\rightarrow$ Ohno–Zagier relation

Let
$$Q_k(\lambda) = \frac{1}{(k-1)!}\sum_i \left(\left(\lambda_i - i + \tfrac{1}{2}\right)^{k-1} - \left(-i+\tfrac{1}{2}\right)^{k-1} \right).$$

Then $\langle Q_k \rangle_q$ is a QMF with
$$\sum_k \left(\lim_{q\to 1} (1-q)^k \langle Q_k \rangle_q \right) z^{k+2} = -1 + \exp\left(\sum_{n\geq 2} \zeta(n) \frac{z^n + (-z)^n}{n} \right).$$

Also
$$Q_k(\lambda) \sim \sum_{i=0}^{k-2} \frac{(-1)^i}{(k-i-1)!} \sum_{m_1 > \cdots > m_{i+1}} m_1^{k-i-1} r_{m_1}(\lambda) \cdots r_{m_{i+1}}(\lambda),$$

from which
$$\lim_{q\to 1} (1-q)^k \langle Q_k \rangle_q = \sum_{i=0}^{k-2} (-1)^i \zeta(k-i, \underbrace{1, \ldots, 1}_{i}).$$

Thm (Ohno–Zagier, '01)
$$\sum_{k\geq 2} \sum_{i=0}^{k-2} (-1)^i \zeta(k-i, \underbrace{1, \ldots, 1}_{i}) z^{k+2} = -1 + \exp\left(\sum_{n\geq 2} \zeta(n) \frac{z^n + (-z)^n}{n} \right).$$

Overview

$$
\begin{array}{ccccc}
\mathbb{Q}^{\mathscr{P}} & \xrightarrow{\langle\,\cdot\,\rangle_q} & \mathbb{Q}[[q]] & \xrightarrow{\lim_{q\to 1}} & \mathbb{R} \\
\cup & & \cup & & \cup \\
P & \longrightarrow & \mathcal{Z}_q & \longrightarrow & \mathcal{Z} \\
\cup & & \cup & & \cup \\
\mathcal{A}\ ,\ \Lambda^{*} & \longrightarrow & \widetilde{M} & \longrightarrow & \mathbb{Q}[j(z)] \\
\cup \quad\ \ \cup & & & & \\
\mathcal{A}_{k,l} \quad \mathbb{Q}_k & & & &
\end{array}
$$

Def (polynomial functions on partitions) Let $f : \mathscr{P} \to \mathbb{Q}$. Then $f \in P$ if $\exists\, p_i \in y_1 \cdots y_i\, \mathbb{Q}[x_1, \dots, x_i, y_1, \dots, y_i]$ for fin. many i s.t.

$$f(\lambda) = p_0 + \sum_i \sum_{m_1 > \dots > m_i > 0} p_i\big(m_1, \dots, m_i, r_{m_1}(\lambda), \dots, r_{m_i}(\lambda)\big).$$

Overview

$$\begin{tikzcd}
\mathbb{Q}^{\mathcal{P}} \arrow[r, "\langle\ \rangle_q"] & \mathbb{Q}[[q]] \arrow[r, "\lim_{q\to 1}"] & \mathbb{R} \\
\mathcal{P} \arrow[u, phantom, "\cup"] \arrow[r] & \mathcal{Z}_q \arrow[u, phantom, "\cup"] \arrow[r, "(ii)"] & \mathcal{Z} \arrow[u, phantom, "\cup"] \\
\mathcal{A},\ \Lambda^* \arrow[u, phantom, "\cup\ (i)"] \arrow[r] & \widetilde{M} \arrow[u, phantom, "\cup"] \arrow[r] & \mathbb{Q}[\zeta(2)] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\mathcal{A}_{k,l} \subset \mathcal{A}$, $\qquad \mathbb{Q}_k \subset \Lambda^*$

no relations between generators (iii)

harmonic & shuffle product (iv)

many relations

harmonic & shuffle product

Thm (Bachmann - vI)

(i) $\mathcal{A}, \Lambda^* \subseteq \mathcal{P}$

(ii) $\forall f \in \mathcal{P}, k \in \mathbb{Z} \qquad \lim (1-q)^k \langle f \rangle_q \in \mathcal{Z} \cup \{\infty\}$

(iii) If $f \in \mathcal{P}$ with $f(\lambda) = 0 \ \forall \lambda \in \mathcal{P}$, then $\rho_i \equiv 0 \ \forall i$

polynomials in def'n of $\mathcal{P}$

(iv) $\mathcal{P}$ is an algebra with an involution, s.t.

$$\langle f \circledast g \rangle_q = \langle f \rangle_q \langle g \rangle_q \qquad \langle \iota(f) \rangle_q = \langle f \rangle_q .$$

product in $\mathcal{P}$

$f \circledast g - \iota(\iota(f) \circledast \iota(g))$ gives double shuffle relations

Thank you!

$$\langle Q_4 \rangle_q = \frac{5E_2^2 + 2E_4}{5760}$$

$$\zeta(4) - \zeta(3,1) + \zeta(2,1,1) = \frac{1}{2}\zeta(2)^2 + \frac{1}{2}\zeta(4) \quad \left(= \frac{7\pi^4}{360} \right)$$