The path model, the quantum Frobenius map and Standard Monomial Theory

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Introduction

The aim of this article is to give an introduction to the theory of path models of representations and its associated bases. The starting point for the theory was a series of articles in which Lakshmibai, Musili and Seshadri initiated a program to construct a basis for the space $H^0(G/B, \mathcal{L}_\lambda)$ with some particularly nice geometric properties. Here we suppose that G is a reductive algebraic group defined over an algebraically closed field k, B is a fixed Borel subgroup, and $\mathcal{L}_\lambda$ is the line bundle on the flag variety G/B associated to a dominant weight. The purpose of the program is to extend the Hodge-Young standard monomial theory for the group $GL(n)$ to the case of any semisimple linear algebraic group and, more generally, to Kac-Moody algebras. Apart from the independent interest in such a construction, the results have important applications to the combinatorics of representations as well as to the geometry of Schubert varieties. Inspired by the classical theory of Young tableaux for representations of $GL_n(\mathbb{C})$, they generalized the notion of a tableau to all classical groups and some exceptional groups and affine Kac-Moody algebras. For the geometric applications note that standard monomial theory provides proofs of the vanishing theorems for the higher cohomology of effective line bundles on Schubert varieties, explicit bases for the rings of invariants in classical invariant theory, a proof of Demazure's conjecture, normality of Schubert varieties, another proof of the good filtration property, determination of the singular locus of Schubert varieties [9], deformation of $SL(n)/B$ into a toric variety [2], etc.

We provide a different access to standard monomial theory which avoids completely all case by case consideration. In the first section we recall the theory of path models. The approach we take here is somewhat different from [16,17]: We start with lattice paths and develop from this point of view the notion of an affine path and the definition of the root operators e_α, f_α. We have provided complete proofs as far as it was made necessary by the new approach. In section 2 and section 7, we discuss some applications of the combinatorics of the path model to character theory (shrinking of characters and crystalline excellent filtration).

In section 4 we construct a basis of $H^0(G/B, \mathcal{L}_\lambda)$ which is indexed in a canonical way by the L-S paths of shape λ. The main tool in our construction is the

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Frobenius map [23] for quantum groups at roots of unity (section 3). Having in mind the applications, one could say that in our approach we replace the "algebraic" Frobenius splitting of Schubert varieties [25,26,27] by the "representation theoretic" Frobenius.

The applications to the geometry of Schubert varieties are discussed in section 5 and 6. For example, the basis is compatible with the restriction map $H^0(G/B,\mathcal{L}_\lambda)\to H^0(X,\mathcal{L}_\lambda)$ to a Schubert variety X, and it has the "standard monomial property". I.e. for λ,μ dominant, there exists a simple combinatorial rule to choose out of the set of all monomials $p_\pi p_\eta\in H^0(G/B,\mathcal{L}_{\lambda+\mu})$ of basis elements $p_\pi\in H^0(G/B,\mathcal{L}_\lambda)$ and $p_\eta\in H^0(G/B,\mathcal{L}_\mu)$, a subset, the *standard monomials*, which forms a basis of $H^0(G/B,\mathcal{L}_{\lambda+\mu})$. Other consequences of the theory are that intersections of unions of Schubert varieties are scheme theoretically reduced, we get a new proof of the Demazure character formula, the vanishing theorem for higher cohomology groups of dominant line bundles and Schubert varieties are projectively normal. In the last section we provide a representation theoretic application of the basis, we provide a new proof of the fact that $H^0(G/B,\mathcal{L}_\lambda)\otimes H^0(G/B,\mathcal{L}_\mu)$ admits a good filtration. All the proofs are characteristic free and work over the ring $\tilde{R}$ obtained from $\mathbb{Z}$ by adjoining all roots of unity. We restrict ourself here for simplicity to the finite dimensional case, but all the results hold, with the obvious adaptions to the infinite dimensional case, for symmetrizable Kac-Moody algebras.

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1. Some combinatorics

Let G be a connected complex reductive algebraic group G. Fix a Borel subgroup $B\subset G$ and a maximal torus $H\subset G$, and let $X=X(H)$ be the weight lattice of H. We denote by $X_\mathbb{R}:=X\otimes_\mathbb{Z}\mathbb{R}$ the corresponding real vector space. We consider now "weights with tails". More precisely, let Π be the set of rectifiable paths in $X_\mathbb{R}$ starting at the origin and ending in an integral weight. Two paths π_1,π_2 are considered as identical if there exists a nondecreasing, surjective, continuous map $\phi:[0,1]\to[0,1]$ (a reparameterization) such that $\pi_1=\pi_2\circ\phi$:

$$\Pi:=\{\pi:[0,1]\longrightarrow X_\mathbb{R}\mid \pi \text{ rectifiable},\ \pi(0)=0,\ \pi(1)\in X\}/(\text{reparameterization}).$$

For simplicity, we will mainly consider the subset $\Pi_\mathbb{Q}\subset\Pi$ of piecewise linear paths having only rational turning points.

The aim of this section is to give an introduction to the combinatorics of the path model of a representation. The notion of an affine path is new and is slightly more general then the notion of L-S paths introduced in [16]. The advantage is that the definition is less technical then the definition of L-S paths. Though we have tried to give complete proofs whenever it seemed appropriate, we have skipped the proofs of those parts which are not used later in the basis construction. We refer to [16,17,20] for detailed proofs.

Example 1. By abuse of notation, we write also λ for the path $\lambda : [0,1] \to X_{\mathbb{R}}$, $t \mapsto t\lambda$, which connects the origin with a weight $\lambda \in X$ by a straight line.

By the concatenation $\pi := \pi_1 * \pi_2$ of two paths π_1, π_2 we mean the path defined by:

$$\pi(t) := \begin{cases} \pi_1(2t), & \textit{if } 0 \leq t \leq 1/2 \\ \pi_1(1) + \pi_2(2t-1), & \textit{if } 1/2 \leq t \leq 1. \end{cases}$$

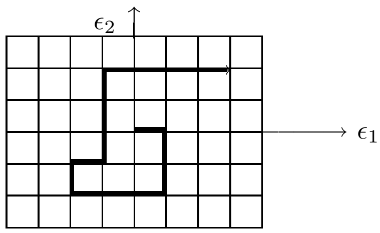
The concatenation is an associative operation, so Π is a monoïde with the trivial path $I : [0,1] \to X_{\mathbb{R}}$, $t \mapsto 0$, as the unit element. Note that the concatenation of paths in $\Pi_{\mathbb{Q}}$ is again a path in $\Pi_{\mathbb{Q}}$, so $\Pi_{\mathbb{Q}} \subset \Pi$ is a submonoïde.

Example 2. *Lattice paths*: Let $H \subset GL_n$ be the subgroup of diagonal matrices, note that H is a maximal torus in GL_n. Denote by ϵ_i the projection of a diagonal matrix onto its i-th entry. The weight lattice $X = X(H)$ is then of rank n with basis the ϵ_i, $i = 1, \ldots, n$, i.e., $X = \mathbb{Z}\epsilon_1 \oplus \ldots \oplus \mathbb{Z}\epsilon_n$ and $X_{\mathbb{R}} \simeq \mathbb{R}^n$.

By a lattice path in $X_{\mathbb{R}}$ we mean a concatenation of the $\pm\epsilon_i$, $i = 1, \ldots, n$, so such a path is of the form $\pi = (\pm\epsilon_{i_1}) * (\pm\epsilon_{i_2}) * \ldots * (\pm\epsilon_{i_r})$. Denote by $L \subset \Pi$ the submonoïde of all lattice paths, and let $L^+ \subset L$ be the submonoïde of lattice paths that are concatenations only of the ϵ_i, $i = 1, \ldots, n$.

The picture below ($n = 2$) shows the image of the lattice path:

$$\pi = \epsilon_1 * (-\epsilon_2) * (-\epsilon_2) * (-\epsilon_1) * (-\epsilon_1) * (-\epsilon_1) * \epsilon_2 * \epsilon_1 * \epsilon_2 * \epsilon_2 * \epsilon_2 * \epsilon_1 * \epsilon_1 * \epsilon_1 * \epsilon_1$$



Example 3. Let $H' \subset SL_n$ be the subgroup of diagonal matrices of determinant 1. As above, let ϵ_i denote the projection of such a matrix onto its i-th entry as well as the corresponding path $\epsilon_i : [0,1] \to X_{\mathbb{R}}$, $t \mapsto t\epsilon_i$, that joins the origin with the weight by a straight line. Recall that $X = \mathbb{Z}\epsilon_1 + \ldots + \mathbb{Z}\epsilon_n$, but the sum is no longer direct because $\epsilon_1 + \ldots + \epsilon_n = 0$.

We denote by $L^+ \subset \Pi$ the submonoïde of all lattice paths of the form $\pi = \epsilon_{i_1} * \epsilon_{i_2} * \ldots * \epsilon_{i_r}$. Note that for $G = SL_n$ every weight in X is the endpoint of some $\pi \in L^+$.

Example 4. Suppose $G = SL_3$. Consider the set $D(2,1) := \{T_1, T_2, \ldots, T_7, T_8\}$ of all semi-standard Young tableaux (i.e., the entries are strictly increasing in the columns and non-decreasing in the rows) of shape $(2,1)$:

$$D(2,1) := \left\{ T_1 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_2 := \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_3 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_4 := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \right.$$

$$\left. T_5 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_6 := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_7 := \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},\ T_8 := \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\}.$$

We associate to a tableau T a lattice path $\pi_T \in L^+$, which is a concatenation of the ϵ_i according to the entries of the tableaux. We read the entries of the tableau row-wise, from the right to the left in each row, and we start with the top row:

$$\begin{array}{llll} \pi_{T_1} := \epsilon_1 * \epsilon_1 * \epsilon_2, & \pi_{T_2} := \epsilon_1 * \epsilon_1 * \epsilon_3, & \pi_{T_3} := \epsilon_2 * \epsilon_1 * \epsilon_2, & \pi_{T_4} := \epsilon_2 * \epsilon_1 * \epsilon_3, \\ \pi_{T_5} := \epsilon_3 * \epsilon_1 * \epsilon_2, & \pi_{T_6} := \epsilon_3 * \epsilon_1 * \epsilon_3, & \pi_{T_7} := \epsilon_2 * \epsilon_2 * \epsilon_3, & \pi_{T_8} := \epsilon_3 * \epsilon_2 * \epsilon_3. \end{array}$$

The endpoint $\pi_T(1)$ of the associated path is the weight of the corresponding tableau.

Example 5. Suppose $G = SL_n$. The general rule to associate a path to a semi-standard Young tableau T is the following: Numerate the boxes $b_1, \ldots, b_\ell$ of the tableau row-wise, from the right to the left in each row, and we start with the top row. Let n_j be the entry of the j-th box. We define the path associated to the tableau T to be the concatenation $\pi_T := \epsilon_{n_1} * \ldots * \epsilon_{n_\ell}$. In this way we can identify the semi-standard Young tableaux with a special class of lattice paths.

The connection between the combinatorics of lattice paths (or rather the word algebra) and tableaux has been studied extensively, see for example [8,13].

Example 6. *Affine paths* Let G be an arbitrary connected complex reductive algebraic group. It turns out that in the general case it is not sufficient to consider only lattice paths. The affine paths can be thought of as a generalization of lattice paths.

We fix a Weyl group invariant scalar product $(\cdot,\cdot)$ on on $X_\mathbb{R}$. Denote by Φ the root system of G, by Φ^+ the subset of positive roots, and for $\beta \in \Phi^+$ let $\beta^\vee := 2\beta/(\beta,\beta)$ be its co-root. We write $\nu \succ \mu$ for two weights $\nu, \mu \in X$ if the difference is a sum of positive roots, and for $\beta \in \Phi$ we write $\beta \succ 0$ if $\beta \in \Phi^+$. For $m \in \mathbb{Z}$ denote by $\mathtt{H}_\beta^m$ the affine hyperplane:

$$\mathtt{H}_\beta^m := \{\nu \in X_\mathbb{R} \mid (\nu, \beta^\vee) = m\}.$$

Denote by $\mathcal{H}$ the union of the $\mathtt{H}_\beta^0$ and by $\check{\mathcal{H}}$ the union of the affine hyperplanes $\mathtt{H}_\beta^m$:

$$\mathcal{H} = \bigcup_{\beta \succ 0} \mathtt{H}_\beta^0, \qquad \check{\mathcal{H}} = \bigcup_{\beta \succ 0, m \in \mathbb{Z}} \mathtt{H}_\beta^m.$$

Recall that we can characterize the simple roots among the positive roots by the following property: $\alpha \succ 0$ is a simple root if and only if $s_\alpha(\gamma) \succ 0$ for all $\gamma \succ 0$, $\gamma \neq \alpha$. In other words: α is simple if and only if for any $\gamma \succ 0$ the property $s_\alpha(\gamma) \prec 0$ implies $\gamma = \alpha$.

Let R be a subset of the set of simple roots. We say that $\beta \in R$ *is simple* if for any $\gamma \in R$ the properties $s_\beta(\gamma) \prec 0$ and $-s_\beta(\gamma) \in R$ implies $\gamma = \beta$.

Let ν, μ be rational weights. We say that μ is obtained from ν by a *simple bending with respect to a positive root* β if $(\nu, \beta^\vee) < 0$, $s_\beta(\nu) = C\mu$ for some $C > 0$, and β is simple (in the sense above) among the positive roots such that $(\nu, \gamma) \leq 0$.

For example, if α is a simple root and $(\nu, \alpha) < 0$, then $(\nu, s_\alpha(\nu))$ is a simple bending.

Suppose $\pi : [0,1] \to X_{\mathbb{R}}$, $\pi \in \Pi_{\mathbb{Q}}$, is a path with rational turning points, so we can find rational weights $\nu_1, \dots, \nu_r$ such that (up to reparameterization) $\pi = \nu_1 * \dots * \nu_r$.

Definition 1. The path π is called an *affine path* if the following conditions are satisfied:

i) All turning points lie in $\check{\mathcal{H}}$.

ii) Suppose $P = \nu_1 + \dots + \nu_s$ is a turning point and $P \notin X$. Then there exist some positive roots $\beta_1, \dots, \beta_q$ such that $P \in \bigcap_{i=1}^{q} \mathtt{H}_{\beta_i}^{m_i}$ for appropriate $m_1, \dots, m_q \in \mathbb{Z}$, and ν_{s+1} is obtained from ν_s by a sequence of simple bendings: $(\nu_s, s_{\beta_1}(\nu_s))$ with respect to β_1, ..., and $(s_{\beta_{q-1}} \dots s_{\beta_1}(\nu_s), \nu_{s+1})$ with respect to β_q.

Roughly speaking, the conditions imply that the path can change its direction only in a point P lying in the set $\check{\mathcal{H}}$, and, if P is not an integral weight, then it is only allowed to change the direction by applying simple bendings. Note that these reflections correspond to roots that are associated to the affine hyperplanes containing P.

A priori, the property of being an affine path depends on the chosen parameterization. If we speak of an affine path $\pi = \nu_1 * \dots * \nu_r$, then we assume always that the path is affine with respect to this parameterization. In particular, all the turning points $P_i = \nu_1 + \dots + \nu_i$ are elements of $\check{\mathcal{H}}$.

To explain the "usefulness" of the simple bending, note the following "geometric" consequence for turning points with respect to simple roots.

Lemma 1. *Let $\pi = \nu_1 * \dots * \nu_r$ be an affine path and denote by $P_i = \nu_1 + \dots + \nu_i$ the i-th turning point. Fix a simple root α and suppose that π changes in P_i its direction relative to α such that either $(\nu_i, \alpha^\vee) < 0$ and $(\nu_{i+1}, \alpha^\vee) \geq 0$, or $(\nu_i, \alpha^\vee) = 0$ and $(\nu_{i+1}, \alpha^\vee) > 0$. Then $P_i \in \mathtt{H}_\alpha^m$ for some $m \in \mathbb{Z}$.*

Proof. This is obvious if $P_i \in X$. We may hence assume that $P_i \notin X$, so P_i satisfies condition *ii)*. Suppose that μ is obtained from ν by a simple bending and $(\nu, \alpha^\vee) < 0$ and $(\mu, \alpha^\vee) \geq 0$, or $(\nu, \alpha^\vee) = 0$ and $(\mu, \alpha^\vee) > 0$. We will show that the bending is of the form $\mu = C s_\alpha(\nu)$. Since ν_{i+1} is obtained from ν_i by a sequence of simple bendings, one of them has to correspond to the reflection s_α. By the definition of an affine path, this implies $P_i \in \mathtt{H}_\alpha^m$ for some $m \in \mathbb{Z}$ and finishes hence the proof.

Let $\beta \succ 0$, $\beta \neq \alpha$ be such that $\mu = s_\beta(\nu)$ is obtained from ν by a simple bending. Suppose first $(\nu, \alpha^\vee) < 0$, and $(\mu, \alpha^\vee) \geq 0$. Since $(\nu, \beta^\vee) < 0$ we can have

$$(\mu, \alpha^\vee) = (s_\beta(\nu), \alpha^\vee) = (\nu - (\nu, \beta^\vee)\beta, \alpha^\vee) = (\nu, \alpha^\vee) - (\nu, \beta^\vee)(\beta, \alpha^\vee) \geq 0$$

only if $(\beta, \alpha^\vee) > 0$. But this implies $s_\beta(\alpha) \prec 0$. It follows that $-s_\beta(\alpha) \succ 0$ and

$$(\nu, -s_\beta(\alpha^\vee)) = -(s_\beta(\nu), \alpha^\vee) = -(\mu, \alpha^\vee) \leq 0,$$

which contradicts the assumption that β is simple in the set of $\gamma \succ 0$ such that $(\nu, \gamma) \leq 0$.

The same arguments apply also to the case $(\nu, \alpha^\vee) = 0$ and $(\mu, \alpha^\vee) > 0$. $\square$

Example 7.

i) If $\mu_1, \dots, \mu_r \in X$ is a sequence of integral weights, then $\pi := \mu_1 * \mu_2 * \dots * \mu_r$ is an affine path. In particular, the lattice paths defined in Example 2 and 3 are affine paths.

ii) If π, η are affine paths, then $\pi * \eta$ is again an affine path. So the set of affine paths forms a submonoïde of Π.

iii) If necessary, then we can introduce more "turning points" on an affine path. Let $\pi := \nu_1 * \nu_2 * \dots * \nu_r$ be an affine path. Suppose $1 \leq i \leq r$ and $c, c' > 0$ are such that $c + c' = 1$ and $P := \nu_1 + \dots + \nu_{i-1} + c\nu_i \in \check{\mathcal{H}}$. Then $\pi' := \nu_1 * \nu_2 * \dots * \nu_{i-1} * c\nu_i * c'\nu_i * \nu_{i+1} * \dots * \nu_r$ and π have the same image, and π' is an affine path: all turning points except P are admissible by assumption. Further, $P \in \check{\mathcal{H}}$, so condition *i)* is satisfied, and *ii)* is trivially satisfied because the path does not change its "direction" in P.

Example 8. A way to produce new paths out of given ones is by stretching them: For $n \in \mathbb{N}$ and $\pi \in \Pi$ let $n\pi$ be the path obtained by stretching π by the factor n, i.e., $(n\pi)(t) := n(\pi(t))$ for $t \in [0, 1]$. Note that if π is an affine path, then $n\pi$ is affine too. Further, if $\eta \in \Pi_{\mathbb{Q}}$ is an arbitrary path, then we can always find an $n \in \mathbb{N}$ such that $n\eta$ is an affine path. For example, we could choose n such that all turning points of $n\eta$ are integral weights.

The evaluation map $\mathrm{Char} : \Pi \to X$, $\pi \mapsto \pi(1)$, is a map of monoïdes:

$$\mathrm{Char}(\pi_1 * \pi_2) = (\pi_1 * \pi_2)(1) = \pi_1(1) + \pi_2(1) = \mathrm{Char}(\pi_1) + \mathrm{Char}(\pi_2).$$

Let $\mathbb{Z}[\Pi]$ be the (non-commutative) $\mathbb{Z}$-algebra generated by Π, and let $\mathbb{Z}[X]$ be the (commutative) group algebra over the weight lattice. We extend the evaluation map to a map of $\mathbb{Z}$-algebras:

$$\mathrm{Char} : \mathbb{Z}[\Pi] \longrightarrow \mathbb{Z}[X], \quad \sum_{i=1}^{k} a_i \pi_i \mapsto \sum_{i=1}^{k} a_i e^{\pi_i(1)}.$$

Example 9. For $G = SL_3$ and $D := D(2, 1)$, let S be the formal sum $\sum_{T \in D} \pi_T \in \mathbb{Z}[\Pi]$ over all semi-standard Young tableaux of shape $(2, 1)$ (Example 4). The image $\sum_{T \in D} e^{\pi_T(1)}$ of S by Char is the character of the adjoint representation of G.

Example 10. Suppose now $G = SL_n$. Recall that the irreducible finite dimensional complex representations of $\mathfrak{g}$ are in bijection with partitions $\mathbf{p} = (p_1, \dots, p_n)$ of length $\leq n$ (i.e. $p_1, \dots, p_n$ is a weakly decreasing sequence of non-negative integers). Let $V(\mathbf{p})$ be the corresponding irreducible representation, and let $D(\mathbf{p})$ be the set of all semi-standard Young tableau of shape $\mathbf{p}$, i.e., the tableau has p_1 boxes in the first row, p_2 boxes in the second row, etc. It is well-known that the combinatorics of tableaux and the representation theory of SL_n are closely related: The character of $V(\mathbf{p})$ is the sum $\sum_{D(\mathbf{p})} e^{\nu(T)}$, where $\nu(T)$ denotes the weight of the tableaux T (= the endpoint of the associated lattice path π_T). So in terms of lattice paths we find: $\mathrm{Char}\, V(\mathbf{p}) = \sum_{T \in D(\mathbf{p})} e^{\pi_T(1)}$

Fix a simple root α. To get character formulas for representations as the sum of endpoints of a set of paths, we define linear operators e_α and f_α on $\mathbb{Z}[\Pi]$. More precisely, these operators are maps $\Pi \to \Pi \cup \{0\}$ which will be extended linearly to all of $\mathbb{Z}[\Pi]$.

Example 11. Suppose $G = SL_n$. We will characterize the paths associated to semi-standard Young tableaux by defining operators on $L^+ \cup \{0\}$. For a simple root $\alpha = \epsilon_i - \epsilon_{i+1}$ and a path $\pi = \epsilon_{j_1} * \ldots * \epsilon_{j_\ell}$ set

$$h_\alpha(t) := \sharp\{k | j_k = i,\ 1 \le k \le t\} - \sharp\{k | j_k = i+1,\ 1 \le k \le t\}, \quad \text{for } 0 \le t \le \ell.$$

Let $m^\pi := \min\{h_\alpha(t) | 0 \le t \le \ell\}$ be the minimal value of the $h_\alpha(t)$, note that $m^\pi \le 0$ because $h_\alpha(0) = 0$. We define now operators e_i, f_i on $L^+ \cup \{0\}$ according to the value m^π.

We set $e_i(0) = f_i(0) := 0$.

If $m^\pi = 0$, then set $e_i(\pi) := 0$. Otherwise fix t minimal such that $h_\alpha(t) = m^\pi$. The minimality of t implies that $j_t = i+1$, so $\pi = \epsilon_{j_1} * \ldots * \epsilon_{j_{t-1}} * \epsilon_{i+1} * \epsilon_{j_{t+1}} * \ldots * \epsilon_{j_\ell}$. We set:

$$e_i(\pi) := \epsilon_{j_1} * \ldots * \epsilon_{j_{t-1}} * \epsilon_i * \epsilon_{j_{t+1}} * \ldots * \epsilon_{j_\ell}.$$

If $m^\pi = h_\alpha(\ell)$, then set $f_i(\pi) := 0$. Otherwise fix t maximal such that $h_\alpha(t) = m^\pi$. The maximality of t implies that $j_{t+1} = i$, so $\pi = \epsilon_{j_1} * \ldots * \epsilon_{j_t} * \epsilon_i * \epsilon_{j_{t+2}} * \ldots * \epsilon_{j_\ell}$. We set:

$$f_i(\pi) := \epsilon_{j_1} * \ldots * \epsilon_{j_{t-1}} * \epsilon_{i+1} * \epsilon_{j_{t+1}} * \ldots * \epsilon_{j_\ell}.$$

For a partition $\mathbf{p} = (p_1, \ldots, p_n)$ set

$$\pi_{\mathbf{p}} := \underbrace{\epsilon_1 * \ldots * \epsilon_1}_{p_1} * \underbrace{\epsilon_2 * \ldots * \epsilon_2}_{p_2} * \ldots * \underbrace{\epsilon_n * \ldots * \epsilon_n}_{p_n},$$

and denote by $B(\mathbf{p}) \subset L^+$ the smallest subset containing $\pi_{\mathbf{p}}$ such that $B(\mathbf{p}) \cup \{0\}$ is stable under the operators e_i, f_i, $1 \le i \le n-1$. We leave it as an exercise for the reader to verify that

$$B(\mathbf{p}) = \{\pi_T \mid T \text{ semi-standard Young tableau of shape } \mathbf{p}\}$$

For the action of the operators on tableaux for other classical groups and related combinatorial problems we refer to [7,12,14,18,22], for the connection to Gelfand-Tsetlin patterns and its generalizations we refer to [1,22].

Example 12. We extend the definition of the operators to affine paths. Let $\pi = \nu_1 * \nu_2 * \ldots * \nu_r$ be an affine path. Fix a simple root α, and let $m \le 0$ be minimal such that $\mathtt{H}_\alpha^m \cap \operatorname{Im} \pi \ne \emptyset$. Recall that any turning point relative to a simple root lies on an affine hyperplane corresponding to this root (Lemma 1), so the image of π lies in the affine halfspace $\mathtt{H}_\alpha^{m,+} := \{\nu \in X_\mathbb{R} \mid (\nu, \alpha^\vee) \ge m\}$.

If $m = 0$, then set $e_\alpha(\pi) := 0$. Otherwise, let j be minimal such that $\nu_1 + \ldots + \nu_j \in \mathtt{H}_\alpha^m$, and let $1 \le i \le j$ be maximal such that $\nu_1 + \ldots + \nu_i \in \mathtt{H}_\alpha^{m+1}$.

(We may assume that such a turning point exists by Example 7, and note that $(\nu_{i+1}, \alpha^\vee), \dots, (\nu_j, \alpha^\vee) < 0$ by Lemma 1.) We define

$$e_\alpha(\pi) := \nu_1 * \dots * \nu_i * s_\alpha(\nu_{i+1}) * \dots * s_\alpha(\nu_j) * \nu_{j+1} * \dots * \nu_r.$$

The definition of the operators f_α is similar. If $m = (\pi(1), \alpha^\vee)$, then we set $f_\alpha(\pi) := 0$. Otherwise let i be maximal such that $\nu_1 + \dots + \nu_i \in \mathtt{H}_\alpha^m$, and let $i \le j \le r$ be minimal such that $\nu_1 + \dots + \nu_j \in \mathtt{H}_\alpha^{m+1}$. (We may assume that such a turning point exists by Example 7, and note that $(\nu_{i+1}, \alpha^\vee), \dots, (\nu_j, \alpha^\vee) > 0$ by Lemma 1.) We define

$$f_\alpha(\pi) := \nu_1 * \dots * \nu_i * s_\alpha(\nu_{i+1}) * \dots * s_\alpha(\nu_j) * \nu_{j+1} * \dots * \nu_r.$$

Note if $e_\alpha(\pi) \ne 0$, then $e_\alpha(\pi)(1) = \pi(1) + \alpha$, and if $f_\alpha(\pi) \ne 0$, then $f_\alpha(\pi)(1) = \pi(1) - \alpha$. In particular, $e_\alpha(\pi), f_\alpha(\pi) \in \Pi_\mathbb{Q} \cup \{0\}$.

Note if $G = SL_n$ and π is a lattice path, then $f_\alpha(\pi)$ respectively $e_\alpha(\pi)$ is the same path as defined in Example 11.

It remains to point out that the new paths (if different from 0) are again affine. We will prove this only for the path $e_\alpha(\pi) = \mu_1 * \dots * \mu_r$, the arguments for $f_\alpha(\pi)$ are similar.

Let $P_\ell = \nu_1 + \dots + \nu_\ell$ and $Q_\ell = \mu_1 + \dots + \mu_\ell$ be the turning points of the path π and $e_\alpha(\pi)$. Let i and j be as in the definition of e_α above. Then $P_\ell = Q_\ell$ for $1 \le \ell \le i$, $P_\ell = Q_\ell - \alpha$ for $j \le \ell \le r$ and Q_ℓ is obtained from P_ℓ by an affine reflection. Since $\check{\mathcal{H}}$ is stable under these operations we see that the $Q_\ell \in \check{\mathcal{H}}$. If the turning point is an integral weight, then nothing is to prove. So suppose in the following that the turning point is not an integral weight.

Further, $(\mu_\ell, \mu_{\ell+1}) = (\nu_\ell, \nu_{\ell+1})$ for $\ell < i$ and $j < \ell$, so $\mu_{\ell+1}$ is obtained from μ_ℓ by a sequence of simple bendings.

For $i < \ell < j$ we have $(\mu_\ell, \mu_{\ell+1}) = (s_\alpha(\nu_\ell), s_\alpha(\nu_{\ell+1}))$. Suppose λ, ν are rational weights such that $(\lambda, \alpha), (\eta, \alpha^\vee) > 0$ or $(\lambda, \alpha), (\eta, \alpha^\vee) < 0$, and $C\nu = s_\beta(\lambda)$ is obtained from λ by a simple bending. It is then easy to see that $s_\alpha(\eta)$ is obtained from $s_\alpha(\lambda)$ by a simple bending with respect to the root $s_\alpha(\beta)$. It follows that $s_\alpha(\nu_{\ell+1})$ is obtained from $s_\alpha(\nu_\ell)$ by a sequence of simple bendings.

Note that $(\mu_i, \mu_{i+1}) = (\nu_i, s_\alpha(\nu_{i+1}))$. Since ν_{i+1} is obtained from ν_i by a sequence of simple bendings and $(\nu_{i+1}, s_\alpha(\nu_{i+1}))$ is a simple bending, it follows that μ_{i+1} is obtained from μ_i by a sequence of simple bendings.

It remains to consider the pair $(\mu_j, \mu_{j+1}) = (s_\alpha \nu_j, \nu_{j+1})$. We know that ν_j is obtained from ν_{j+1} by a sequence of simple bendings, let $\beta_1, \dots, \beta_q$ be the corresponding positive roots. Since $(\nu_j, \alpha^\vee) \le 0$ and $(\nu_{j+1}, \alpha^\vee) > 0$, we have seen in the proof of Lemma 1 that at least one of the β_ℓ has to be equal to α. Suppose that $1 \le \ell \le q$ is minimal with this property, one sees then easily that the roots $s_\alpha(\beta), \dots, s_a(\beta_{\ell-1}), \beta_{\ell+1}, \dots, \beta_q$ provide a sequence of simple bendings for the pair $(s_\alpha \nu_j, \nu_{j+1})$. $\square$

Since all turning points relative to the direction of a simple root lie on an affine hyperplane $\mathtt{H}_\alpha^m$, it follows easily that if π, η are affine paths, then $e_\alpha(\pi * \eta) = (e_\alpha \pi) * \eta$ or $\pi * (e_\alpha \eta)$. Similarly, $f_\alpha(\pi * \eta) = (f_\alpha \pi) * \eta$ or $\pi * (f_\alpha \eta)$. More precisely, a simple calculation shows that:

Lemma 2. *If π, η are affine paths, then $e_\alpha(\pi * \eta) = \pi * (e_\alpha \eta)$ if there exists an $n > 0$ such that $e_\alpha^n \eta \neq 0$ but $f_\alpha^n \pi = 0$, and it is equal to $(e_\alpha \pi) * \eta$ otherwise. Similar, $f_\alpha(\pi * \eta) = (f_\alpha \pi) * \eta$ if there exists an $n > 0$ such that $f_\alpha^n \pi \neq 0$ but $e_\alpha^n \eta = 0$, and it is equal to $\pi * (f_\alpha \eta)$ otherwise.*

We generalize the operators now to arbitrary piecewise linear paths. The definition of these operators have been inspired by the work of Kashiwara on crystal bases, see [4,5]. Note that the operators defined on affine paths "commute" with stretching, i.e. $n\big(f_\alpha(\pi)\big) = f_\alpha^n(n\pi)$ respectively $n\big(e_\alpha(\pi)\big) = e_\alpha^n(n\pi)$, and with "shrinking", i.e., if $\frac{1}{n}\pi$ is also an affine path, then $\frac{1}{n}\big(f_\alpha^n(\pi)\big) = f_\alpha(\frac{1}{n}\pi)$ and $\frac{1}{n}\big(e_\alpha^n(\pi)\big) = e_\alpha(\frac{1}{n}\pi)$.

We have seen that if $\pi \in \Pi_{\mathbb{Q}}$, then we can always find an n such that $n\pi$ is affine. We define

$$f_\alpha(\pi) := \frac{1}{n} f_\alpha^n(n\pi), \quad e_\alpha(\pi) := \frac{1}{n} e_\alpha^n(n\pi).$$

Since on affine paths stretching/shrinking commutes with the operators, the definition is independent on the choice of n.

We provide now a different description of the operators, we leave it as an exercise to verify that the definitions coincide. For a path $\pi \in \Pi_{\mathbb{Q}}$ let h^π be the function defined by:

$$h^\pi : [0,1] \to \mathbb{R}, \quad t \mapsto (\pi(t), \alpha^\vee),$$

and let $m^\pi := \min h^\pi$ be the minimal value attained by the function h^π. If $h^\pi(1) - m^\pi \geq 1$, then fix $t_0 \in [0,1]$ maximal such that $h^\pi(t_0) = m^\pi$ and let t_1 be minimal such that $h^\pi(t) \geq m^\pi + 1$ for $t \geq t_1$. Denote by $\phi_+ : [0,1] \to \mathbb{R}$ the function:

$$\phi_+(t) := \begin{cases} 0, & \text{if } t \in [0, t_0]; \\ \min\{h^\pi(s) - m^\pi \mid s \in [t, t_1]\}, & \text{if } t \in [t_0, t_1]; \\ 1, & \text{if } t \in [t_1, 1]. \end{cases}$$

FIGURE 1. The functions $h^\pi(t)$ and $\phi_+(t)$

Definition 2. If $h^\pi(1) - m^\pi < 1$, then set $f_\alpha(\pi) := 0$. Otherwise let $f_\alpha(\pi) \in \Pi_\mathbb{Q}$ be the path defined by $f_\alpha(\pi)(t) := \pi(t) - \phi_+(t)\alpha$.

The definition of the operator e_α is similar. If $m^\pi \leq -1$, then fix $t_0 \in [0,1]$ maximal such that $h^\pi(t) \geq m^\pi + 1$ for $t \leq t_0$ and let t_1 be minimal such that $h^\pi(t) = m^\pi$. Denote by $\phi_- : [0,1] \to \mathbb{R}$ the function:

$$\phi_-(t) := \begin{cases} 0, & \text{if } t \in [0, t_0]; \\ \max\{m^\pi - h^\pi(s) + 1 \mid s \in [t_0, t]\}, & \text{if } t \in [t_0, t_1]; \\ 1, & \text{if } t \in [t_1, 1]. \end{cases}$$

Definition 3. If $m^\pi > -1$, then set $e_\alpha(\pi) := 0$. Otherwise let $e_\alpha(\pi) \in \Pi_\mathbb{Q}$ be the path defined by $e_\alpha(\pi)(t) := \pi(t) + \phi_-(t)\alpha$.

A chamber $C \subset X_\mathbb{R}$ is a connected component of $X_\mathbb{R} - \mathfrak{H}$, and an alcove A is a connected component of $X_\mathbb{R} - \check{\mathcal{H}}$. The closure $\overline{C}$ of C respectively the closure $\overline{A}$ of A is taken in the usual topology on $\mathbb{R}^n$. The closure of the chamber $\overline{C_0} := \{\nu \in X_\mathbb{R} \mid (\nu, \beta^\vee) > 0 \, \forall \beta \succ 0\}$ is called the dominant Weyl chamber. The following properties follow easily from the definition of the operators:

Lemma 3.
a) If $e_\alpha\pi \neq 0$, then $e_\alpha(\pi)(1) = \pi(1) + \alpha$ and $f_\alpha(e_\alpha\pi) = \pi$.
b) If $f_\alpha\pi \neq 0$, then $f_\alpha(\pi)(1) = \pi(1) - \alpha$ and $e_\alpha(f_\alpha\pi) = \pi$.
c) Let ρ be half the sum of the positive roots. Then $e_\alpha\pi = 0$ for all simple roots if and only if $\rho + \operatorname{Im}\pi$ (the image of π shifted by ρ) is completely contained in the interior C_0 of the dominant Weyl chamber $\overline{C_0}$.
d) Let m, n be maximal such that $e_a^m\pi \neq 0$ resp. $f_a^n\pi \neq 0$. Then $n - m = (\pi(1), \alpha^\vee)$, and

$$m = \max\{a \in \mathbb{Z} \mid a \leq |m^\pi|\}, \quad n = \max\{a \in \mathbb{Z} \mid a \leq (\pi(1), \alpha^\vee) - m^\pi\}.$$

e) The stretching of paths commutes with the operators, i.e., $n(e_\alpha\pi) = e_\alpha^n(n\pi)$ and $n(f_\alpha\pi) = f_\alpha^n(n\pi)$.

Extend the operators linearly to the algebra $\mathbb{Z}[\Pi_\mathbb{Q}]$. For a path π denote by $B(\pi) \subset \Pi_\mathbb{Q}$ the set of all paths one gets from π by applying the operators. The $\mathbb{Z}$-span $\mathbb{Z}B(\pi) \subset \mathbb{Z}[\Pi_\mathbb{Q}]$ is then the minimal submodule which is stable under the operators and contains π.

Example 13. Recall that we denote by $\mu \in X$ also the straight line joining the origin with the weight μ. For a simple root α set $n := (\mu, \alpha^\vee)$. It is easy to see that if $n \geq 0$, then $e_\alpha(\mu) = 0$ and $f_\alpha^n(\mu) = s_\alpha(\mu)$, and if $n < 0$, then $f_\alpha(\mu) = 0$ and $e_\alpha^{-n}(\mu) = s_\alpha(\mu)$. As a consequence we find all the paths $w(\mu)$, $w \in W$, in $B(\mu)$.

Example 14. By part e) of the lemma we have a well defined map $B(\pi) \hookrightarrow B(n\pi)$, $\eta \mapsto n\eta$. We can hence identify $B(\pi)$ with the subset of $B(n\pi)$ obtained by applying only n-th powers of the operators to $n\pi$.

Proposition 1. *The set $B(\pi)$, $\pi \in \Pi_{\mathbb{Q}}$, is finite.*

Proof. Let $\pi : [0,1] \to X_{\mathbb{R}}$, $\pi = \nu_1 * \ldots * \nu_p$ be an affine path. By Example 7 we may add turning points if necessary and assume that all "parts" of the path are completely contained in the closure of some alcove. I.e., we may assume that for all $j = 1, \ldots, q$, there exists a (not necessarily unique) alcove A_j such that the image of the path $t \mapsto t\nu_j$, shifted by the weight $\nu_1 + \ldots + \nu_{j-1}$, is contained in $\overline{A}_j$. In particular, $\operatorname{Im}\pi \subset \bigcup_j \overline{A}_j$. So we can think of the path as a sequence of alcoves A_j together with a segment $S_j \subset \overline{A}_j$ joining two points $p_1, p_2 \in \overline{A}_j - A_j$.

Suppose $f_\alpha(\pi) = \mu_1 * \ldots * \mu_p \neq 0$. If we want to choose alcoves for this path with the same properties as above, then, by the construction of the operators, we know that there exist i, j such that we may take for $f_\alpha(\pi)$ the alcoves $\overline{A}_\ell$ for $1 \leq \ell \leq i$, the alcoves $\overline{A}_\ell + \alpha$ (i.e., shifted by the root α) for $j \leq \ell \leq p$, and $s_{\alpha,m}(\overline{A}_\ell)$ for $i < \ell \leq j$ (and appropriate m). Here $s_{\alpha,m}$ denotes the affine reflection at the affine hyperplane $\mathtt{H}_\alpha^m$. The same arguments apply also to a path of the form $e_\alpha(\pi)$.

It follows that we can view the set $B(\pi)$ as a subset of the set of paths we get by rearranging a given finite set of "marked" alcoves. Recall that a face F of an alcove A is the intersection $\overline{A} \cap \mathtt{H}_{\beta_1}^{m_1} \cap \ldots \cap \mathtt{H}_{\beta_r}^{m_r}$ of $\overline{A}$ with some affine hyperplanes $\mathtt{H}_{\beta_1}^{m_1}, \ldots, \mathtt{H}_{\beta_r}^{m_r} \in \check{\mathcal{H}}$. So applying the operators is like rearranging the alcoves such that they are still joined along common faces, and the segments of two such joined alcoves have each an endpoint in the common face, and these endpoints match.

The moves which are allowed for the "rearrangements" are affine reflections and translation with respect to simple roots. Since these rearrangements have to be connected to the origin (i.e., the first segment has the origin as one of the endpoints), we see that there are only a finite number of possibilities. In particular, $B(\pi)$ is a finite set.

Now suppose $\pi \in \Pi_{\mathbb{Q}}$ is an arbitrary path. In Example 14 we have seen that we can identify $B(\pi)$ with a subset of $B(n\pi)$. If we choose n such that $n\pi$ is affine (Example 8), then $B(n\pi)$ is finite by the arguments above, and hence $B(\pi)$ is a finite set. $\square$

Remark 1. The proof shows that we can think of the "affine" paths as a kind of domino game in the affine space $X_{\mathbb{R}}$. The pieces are closed alcoves together with mark: a segment joining two points in $\overline{A} - A$. One rule to join two pieces along a face is of course that the the endpoints of the segments should match, but, if the "direction" of the segment changes, then one has to assure that the change is obtained by a sequence of "allowed" simple bendings. This definition assures that the root operators f_α, e_α translate in this language into operators on a given arrangement of marked "pieces", i.e., the alcoves.

Lemma 4. *For $\pi \in \Pi_{\mathbb{Q}}$ there exists an $\eta \in \Pi_{\mathbb{Q}}$ such that $\rho + \operatorname{Im}\eta \subset C_0$ and $B(\eta) = B(\pi)$. If π is affine, then there exists an affine η such that $\operatorname{Im}\eta \subset \overline{C_0}$ and $B(\eta) = B(\pi)$*

Proof. We know by Lemma 3 a),b): $B(\pi) = B(\eta)$ for $\eta \in B(\pi)$, so we may replace $B(\pi)$ by $B(e_\alpha\pi)$ if $e_\alpha\pi \neq 0$. Since $B(\pi)$ is finite, we may hence assume that

$e_\alpha\pi = 0$ for all simple roots. By Lemma 3, this is equivalent to $\operatorname{Im}\eta + \rho \subset C_0$. If π is affine, then $e_\alpha(\pi) = 0$ means that $\operatorname{Im}\pi \cap \mathtt{H}_\alpha^{-1} = \emptyset$. So $\operatorname{Im}\pi$ lies in the affine halfspace $\mathtt{H}_\alpha^{0,+}$ (see Example 12), and hence $\operatorname{Im}\eta \subset \overline{C_0}$. $\square$

Theorem 1.

a) Let $B \subset \Pi$ be a finite subset such that the span $\mathbb{Z}B \subset \mathbb{Z}[\Pi]$ is stable under the root operators, and let $\pi_1, \dots, \pi_s \in B$ be the paths such that $\operatorname{Im}\pi_j + \rho \subset C_0$. The sum $\operatorname{Char} B := \sum_{\eta\in B(\pi)} e^{\eta(1)}$ *is the sum of the characters of the finite dimensional irreducible representations of G of highest weight $\lambda_1 := \pi_1(1), \dots, \lambda_s := \pi_s(1)$.*

b) For a dominant weight $\lambda \in X^+$ denote by $\lambda : [0,1] \to X_\mathbb{R}$ also the path $t \mapsto t\lambda$. Then $\operatorname{Char} B(\lambda)$ *is the character of the irreducible representation $V(\lambda)$ of highest weight λ.*

Remark 2. The part b) of the theorem above holds in much more generality, but in the rest of this paper we need only the weaker statement above. Note that b) is true for any set of paths of the form $B(\pi)$, where π is a path that such that its image is completely contained in the dominant Weyl chamber (see [17] for a proof).

Proof. By the Weyl character formula we have to show: ($\epsilon(w) :=$ the sign of w)

$$\sum_{\pi\in B} e^{\pi(1)} = \sum_{i=1}^{s} \operatorname{Char} V(\lambda_i) = \sum_{i=1}^{s} \big(\sum_{w\in W} \epsilon(w) e^{w(\lambda_i+\rho)}\big)/\big(\sum_{w\in W} \epsilon(w) e^{w(\rho)}\big).$$

To prove the latter is equivalent to prove that

$$\sum_{\pi\in B, w\in W} \epsilon(w) e^{\pi(1)+w(\rho)} = \sum_{i=1}^{s} \big(\sum_{w\in W} \epsilon(w) e^{w(\lambda_i+\rho)}\big).$$

The terms on both sides are semi-invariant with respect to the action on the Weyl group on $\mathbb{Z}[X]$. So to prove the equality above, it is sufficient to compare the terms corresponding to dominant weights:

$$\sum_{\pi\in B, w\in W, \pi(1)+w(\rho)\in X^+} \epsilon(w) e^{\pi(1)+w(\rho)} = \sum_{i=1}^{s} e^{\lambda_i+\rho}.$$

Let B' be the set of pairs (π, w) such that $\pi \in B$ and $w \in W$ are such that $w(\rho) + \pi(1) \in X^+$ and $(\pi, w) \neq (\pi_i, id)$ for some i. In other words: B' is the set of pairs (π, w) occurring in the sum on the left above which have the property that $\operatorname{Im}\pi + w(\rho)$ is not completely contained in the interior of the dominant Weyl chamber. So to prove part a) is equivalent to prove that $\sum_{(\pi,w)\in B'} \epsilon(w) e^{\pi(1)+w(\rho)} = 0$.

Let $p \in \overline{C}_0 - C_0$ be a point lying on a face of the dominant Weyl chamber. Denote by $B'(p)$ the subset of paths $(\pi, w) \in B'$ such that $\pi(t_0) + w(\rho) = p$ for some $t_0 \in [0,1]$ and $\pi(t) + w(\rho)$ lies in the interior of the dominant Weyl chamber for $t_0 < t \leq 1$. Since B' is the disjoint union of such $B'(p)$, to prove $a)$ it suffices to prove $\sum_{(\pi,w)\in B'(p)} \epsilon(w)e^{\pi(1)+w(\rho)} = 0$ for such a point p.

We define an involution I on $B'(p)$. For $(\pi, w) \in B'(p)$ let $t_0 \in [0,1]$ be such that $\pi(t_0) + w(\rho) = p$ and $\pi(t) + w(\rho)$ lies in the interior of the dominant Weyl chamber for $t_0 < t \leq 1$. Let α be a simple root such that $(p, \alpha^\vee) = 0$ and set $n := (w(\rho), \alpha^\vee)$. Suppose first $n > 0$. Since $(w(\rho), \alpha^\vee) + (\pi(t_0), \alpha^\vee) = (p, \alpha^\vee) = 0$, we know that the minimum of the function $t \to (\pi(t), \alpha^\vee)$ is smaller or equal to $-n$. By Lemma 3 this implies that $\pi' := e_\alpha^n(\pi) \neq 0$. Consider the pair $(\pi', s_\alpha w)$. Since $\pi(t) + w(\rho)$ lies in the interior of the dominant Weyl chamber for $t_0 < t \leq 1$, by the definition of the operator e_α we know that $\pi'(t) = \pi(t) + n\alpha$ for $t \geq t_0$ and hence

$$w(\rho) + \pi(t) = s_\alpha w(\rho) + \pi'(t) \quad \text{for } t \geq t_0.$$

In particular, $(\pi', s_\alpha w) \in B'(p)$. So for $n > 0$ we define $I(\pi, w) := (\pi', s_\alpha w)$.

If $n < 0$, then similar arguments show that $I(\pi, w) := (f_\alpha^{-n}\pi, s_\alpha w) \in B'(p)$ and $w(\rho) + \pi(t) = s_\alpha w(\rho) + f_\alpha^{-n}\pi(t)$ for $t \geq t_0$. It is easy to check that I is an involution. Since $\epsilon(w)e^{\pi(1)+w(\rho)} + \epsilon(s_\alpha w)e^{\pi'(1)+s_\alpha w(\rho)} = 0$ for $(\pi', s_\alpha w) = I(\pi, w)$, it follows that $\sum_{(\pi,w)\in B'(p)} \epsilon(w)e^{\pi(1)+w(\rho)} = 0$.

It remains to prove part $b)$. By $a)$ we have to show that the only path $\pi \in B(\lambda)$ such that $\operatorname{Im}\pi + \rho \subset C_0$ is the path $\pi = \lambda$. Since λ is an affine path (Example 7), all paths in $B(\lambda)$ are affine.

Let $\eta = \mu_1 * \ldots * \mu_r \in B(\lambda)$ be an arbitrary path. For all $i = 1, \ldots, r$ we can find an element $w_i \in W/W_\lambda$ such that $\mu_i = c_i w_i(\lambda)$. Suppose $w_1 \geq \ldots \geq w_r$. It is easy to check that then the corresponding Weyl group elements for $f_\alpha\eta, e_\alpha\eta$ have the same property. Since the starting path $\lambda = id(\lambda)$ has obviously this property, it follows that all elements in $B(\lambda)$ have the property that the associated sequence of elements in W/W_λ is decreasing in the Bruhat order.

We have seen in the proof of Lemma 4 that the condition $\operatorname{Im}\pi + \rho \subset C_0$ implies for an affine path in fact $\operatorname{Im}\pi \subset \overline{C_0}$. Suppose $\pi = \nu_1 * \ldots * \nu_p \in B(\lambda)$, then $\operatorname{Im}\pi \subset \overline{C_0}$ implies that $C\nu_1 = \lambda$ for some $C > 0$. But this implies for the path π that the associated sequence of elements in W/W_λ starts with $w_1 = id$, so $w_i = id$ for all i and hence $\pi = \lambda$. $\square$

As noted above, the second part holds more generally for arbitrary paths having its image in the dominant Weyl chamber. Let now π, η be affine paths having its image in the dominant Weyl chamber. Denote by $\lambda := \pi(1)$ and $\mu := \eta(1)$ the endpoints. The set of all concatenations of paths in $B(\pi)$ and $B(\eta)$ is denoted by $B(\pi) * B(\eta)$. By Lemma 2, $B(\pi) * B(\eta) \cup \{0\}$ is stable under the root operators. So we get:

Proposition 2. *$B(\pi)*B(\eta) = \bigcup_{\eta'} B(\pi*\eta')$, where η' is running over all elements in $B(\eta)$ such that the image of $\pi * \eta'$ is contained in the dominant Weyl chamber.*

The character formula implies the following generalization of the Littlewood-Richardson rule:

Corollary 1. $V(\lambda) \otimes V(\mu) \simeq \bigoplus_{\eta'} V(\lambda + \eta'(1))$, *where* η' *is running over all elements in* $B(\eta)$ *such that the image of* $\pi * \eta'$ *is contained in the dominant Weyl chamber.*

We have already pointed out in Remark 2 that the character formula holds for all sets of the form $B(\pi)$, where $\pi \in \Pi_{\mathbb{Q}}$ is such that its image is contained in the dominant Weyl chamber. This independence on the chosen path has in fact a much deeper background. Denote by $\mathcal{G}(\pi)$ the associated colored directed graph having as vertices the set $B(\pi)$, and we put an arrow $\eta \xrightarrow{\alpha} \eta'$ with color a simple root α between two paths $\eta, \eta' \in B(\pi)$ if $f_\alpha(\eta) = \eta'$. A proof of the following theorem can be found in [17] or [20]. The theorem implies that the graph is isomorphic to the crystal graph of the corresponding representation (see [6] or [3]).

Theorem 2. *Suppose* $\pi, \pi' \in \Pi_{\mathbb{Q}}$ *are such that* $\operatorname{Im}\pi, \operatorname{Im}\pi'$ *are contained in the dominant Weyl chamber. The map* $\pi \to \pi'$ *extends to an isomorphism of colored directed graphs* $\mathcal{G}(\pi) \to \mathcal{G}(\pi')$ *if and only if* $\pi(1) = \pi'(1)$.

2. Contracting characters

There are many ways to transform the path model of a representation such that the new set of paths is stable under the operators and hence corresponds to a character of a representation. One method is that of contracting or shrinking the paths. This section is not really needed in the following, but the results are interesting on its own and have been the motivation for the construction of the basis in the next section.

For a dominant weight $\lambda \in X^+$ let $V(\lambda)$ be the corresponding irreducible complex representation of G. Fix a natural number $\ell \in \mathbb{N}$, and let $\rho \in X$ be half the sum of the positive roots. By the *contracted character* $\operatorname{Char} V(\lambda)^{\frac{1}{\ell}}$ we mean the (Weyl group invariant) sum

$$\operatorname{Char} V(\lambda)^{\frac{1}{\ell}} := \sum_{\mu \in \ell X} \dim V_\mu(\lambda) e^{\frac{\mu}{\ell}}.$$

Proposition 3. $\operatorname{Char} V(\lambda)^{\frac{1}{\ell}}$ *is a nonnegative linear combination of characters of irreducible representations:*

$$\operatorname{Char} V(\lambda)^{\frac{1}{\ell}} = \sum_{\nu \in X^+} m^{\nu}_{\lambda,\ell} \operatorname{Char} V(\nu).$$

The multiplicity $m^{\nu}_{\lambda,\ell}$ *with which the character of* V_ν *occurs in the sum above is equal to the multiplicity of* $V((\ell-1)\rho + \ell\nu)$ *in the tensor product* $V((\ell-1)\rho) \otimes V(\lambda)$.

Proof. As before, denote by λ also the path joining the origin with λ by a straight line, and let $B(\lambda)$ be the corresponding set of paths obtained by applying the root operators. The character of $V(\lambda)$ is given by

$$\operatorname{Char} V(\lambda) = \sum_{\eta \in B(\lambda)} e^{\eta(1)}.$$

If $\eta(1) \in \ell X$, then the path $(\eta/\ell) : [0,1] \to X_{\mathbb{R}}$, $t \mapsto \eta(t)/\ell$, is again an element of $\Pi_{\mathbb{Q}}$ (but not necessarily affine). Denote by $B(\lambda)^\ell$ the set of paths $\eta \in B(\lambda)$ such that $\eta(1) \in \ell X$, and let

$$\check{B}(\lambda) := \{\eta/\ell \mid \eta \in B(\lambda)^\ell\}$$

be the set of contracted paths. We have obviously: $\mathrm{Char}\, V(\lambda)^{\frac{1}{\ell}} = \sum_{\eta \in \check{B}(\lambda)} e^{\eta(1)}$.

The root operators commute with "stretching": For all $k \in \mathbb{N}$ and all $\eta \in \Pi$ we have $f_\alpha(k\eta) = f_\alpha^k(\eta)$ and $e_\alpha(k\eta) = e_\alpha^k(\eta)$. Since $B(\lambda)^\ell \cup \{0\}$ is stable under the ℓ-th powers of the root operators, it follows that the set $\check{B}(\lambda) \cup \{0\}$ is stable under the root operators. The character formula implies:

$$\sum_{\eta \in \check{B}(\lambda)} e^{\eta(1)} = \sum_{\eta \in \check{B}(\lambda)_+} \mathrm{Char}\, V(\eta(1)),$$

where $\check{B}(\lambda)_+ \subset \check{B}(\lambda)$ is the subset of paths η such that $\mathrm{Im}\, \eta + \rho$ is contained in C_0. The latter is equivalent to say that $\mathrm{Im}(\ell\eta) + \ell\rho$ is contained in C_0. Since $\ell\eta$ is affine, this is equivalent to say that $\mathrm{Im}(\ell-1)\rho * \ell\eta$ is contained in $\overline{C}_0$. By the tensor product formula this implies that the multiplicity $m^\nu_{\lambda,\ell}$ is equal to the multiplicity of $V((\ell-1)\rho + \ell\nu)$ in $V((\ell-1)\rho) \otimes V(\lambda)$. $\square$

We describe now a different version of the contraction procedure which, as we will see later, has a representation theoretic interpretation. We fix a numeration of the simple roots $\alpha_1, \ldots, \alpha_n$, let $A = (a_{i,j})_{1 \le i,j \le n}$ be the Cartan matrix of $\mathfrak{g}$ and denote by A^t the transposed matrix. Fix $\underline{d} = (d_1, \ldots, d_n)$, $d_i \in \mathbb{N}$, minimal such that the matrix $(d_i a_{i,j})$ is symmetric. We denote by d the smallest common multiple of the d_j, and we set $\overline{d} = (\overline{d}_1, \ldots, \overline{d}_n)$, where $\overline{d}_i := d/d_i$. The form $(\cdot,\cdot)$ is chosen such that $(\alpha_i^\vee, h) := \langle \alpha_i, h\rangle / d_i$.

The transposed matrix $A^t = (\overline{a}_{i,j})$, $\overline{a}_{i,j} := a_{j,i}$ is the Cartan matrix of the root system Φ^t dual to the root system Φ of $\mathfrak{g}$. To realize Φ^t as a root system in $\mathfrak{H}^*$, note that the $\overline{d}_i$ are minimal with the property that $(\overline{d}_j \overline{a}_{i,j})$ is a symmetric matrix. The triple $(\mathfrak{H}, \Gamma, \Gamma^\vee)$ defined by $\gamma_i := \alpha_i/d_i \in \mathfrak{H}^*$ and $\gamma_i^\vee := d_i \alpha_i^\vee \in \mathfrak{H}$ is a realization of A^t, i.e., $\gamma_1, \ldots, \gamma_n$ is a basis of a root system isomorphic to Φ^t, and the $\gamma_i^\vee$ form a set of co-roots. Denote by X^t the weight lattice for the dual root system:

$$X^t = \{\lambda \in \mathfrak{H}^* \mid \forall i : \langle \lambda, \gamma_i^\vee\rangle \in \mathbb{Z}\} = \{\lambda \in \mathfrak{H}^* \mid \forall i : d_i \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z}\}.$$

It follows immediately from the definition: $dX^t \subset X \subset X^t$.

Denote by $\mathfrak{g}$ the Lie algebra of G. Suppose now $\lambda \in X^t$ is a dominant weight for the semisimple Lie algebra $\mathfrak{g}^t$ associated to the Cartan matrix A^t, let $V(\lambda)^t$ be the corresponding complex irreducible highest weight representation. Fix a natural number l divisible by d and set $\overline{\ell} := \ell/d$. Let $\rho^t \in X^t$ be half of the positive roots for the root system of $\mathfrak{g}^t$.

Proposition 4. $\mathrm{Char}(V(\lambda)^t)^{\overline{\ell}} := \sum_{\mu\in\overline{\ell}X} \dim V_\mu(\lambda)^t e^{\frac{\mu}{\overline{\ell}}}$ *is a nonnegative linear combination of characters of irreducible* $\mathfrak{g}$*-representations:*

$$\mathrm{Char}(V(\lambda)^t)^{\overline{\ell}} = \sum_{\nu\in X^+} m^\nu_{\lambda,\ell} \,\mathrm{Char}\, V(\nu).$$

The multiplicity $m^\nu_{\lambda,\ell}$ *is equal to the multiplicity of* $V^t(\overline{\ell}\rho - \rho^t + \overline{\ell}\nu)$ *in the tensor product of* $\mathfrak{g}^t$*-representations* $V^t(\overline{\ell}\rho - \rho^t)\otimes V^t(\lambda)$*.*

Proof. We attach to a $\mathfrak{g}^t$-object a ^{t} to distinguish it from the corresponding $\mathfrak{g}$-object. Note that the dominant Weyl chamber in $X_{\mathbb{R}} = X^t_{\mathbb{R}}$ is for both the same. Let λ be the path that joins the weight λ with the origin by a straight line. Denote by B^t be the set of paths obtained from λ by applying the root operators $f_{\gamma_i}, e_{\gamma_i}$. The character of $V^t(\lambda)$ is equal to $\sum_{\eta\in B^t} e^{\eta(1)}$.

Denote by $B^{\overline{\ell}}$ the subset of B^t of paths ending in a point of the lattice $\overline{\ell}X$, and let $\check{B}$ be the set of paths $\{\eta/\overline{\ell} \mid \eta\in B^{\overline{\ell}}\}$. We will show that the set $\check{B}\cup\{0\}$ is stable under the root operators $f_{\alpha_i}, e_{\alpha_i}$.

Since $\gamma_i^\vee = d_i\alpha_i^\vee$ and $\gamma_i = \alpha_i/d_i$, it follows by the definition of the operators that $f^{d_i}_{\gamma_i} = f_{\alpha_i}$. By assumption we know that ℓ is divisible by d and hence $f^{d_i\ell/d}_{\gamma_i} = f^{\overline{\ell}}_{\alpha_i}$. If $\eta\in B^{\overline{\ell}}$, and $f^{\overline{\ell}}_{\alpha_i}(\eta)\neq 0$, then $f^{\overline{\ell}}_{\alpha_i}(\eta) = f^{d_i\ell/d}_{\gamma_i}(\eta)\in B^t$, and the endpoint of the path is $\eta(1)-\overline{\ell}\alpha_i\in\overline{\ell}X$. This proves: $f^{\overline{\ell}}_{\alpha_i}(\eta)\in B^{\overline{\ell}}$, and therefore: $\check{B}\cup\{0\}$ is stable under the root operators f_{α_i}, $i=1,\dots,n$. The proof for the operator e_{α_i} is similar.

Now the same arguments as in the proof of Proposition 3 apply. We have to look for those paths $\eta\in\check{B}$ such that $\rho + \mathrm{Im}\,\eta$ is in contained in C_0, but this condition is equivalent to the condition that $\overline{\ell}\rho + \mathrm{Im}(\overline{\ell}\eta)$ stays in C_0. Since $\overline{\ell}\eta$ is affine (for $\mathfrak{g}^t$), this condition is equivalent to $\mathrm{Im}\big((\overline{\ell}\rho-\rho^t)*\overline{\ell}\eta\big)$ is contained in $\overline{C}_0$. The same arguments as above prove hence the multiplicity formula. $\square$

3. The quantum Frobenius map

We use the same notation as in section 2. For a dominant weight $\lambda\in X^{t,+}$ let $N(\lambda)$ be the Weyl module for the quantum group $U_{\mathbf{v}}(\mathfrak{g}^t)$ at a 2ℓ-th root of unity. The subspace $\bigoplus_{\mu\in\overline{\ell}X} N(\lambda)_\mu$ admits in a natural way a $U(\mathfrak{g})$-action, which gives a representation theoretic interpretation of Proposition 4. We assume throughout the rest of the article that ℓ is divisible by $2d$.

Let $U_q(\mathfrak{g}^t)$ be the quantum group associated to $\mathfrak{g}^t$ over the field $\mathbb{Q}(q)$, with generators $E_{\gamma_i}, F_{\gamma_i}, K_{\gamma_i}$ and $K^{-1}_{\gamma_i}$. We use the usual abbreviations

$$[n]_i := \frac{q^{\overline{d}_i n} - q^{-\overline{d}_i n}}{q^{\overline{d}_i} - q^{-\overline{d}_i}},\ [n]_i! := [1]_i\cdots[n]_i,\ \begin{bmatrix} n\\ m\end{bmatrix}_i := \frac{[n]_i!}{[m]_i![n-m]_i!},$$

where the latter is supposed to be zero for $n<m$. We write sometimes $E_i, K_i,\dots$ for $E_{\gamma_i}, K_{\gamma_i},\dots$ and

$$q_i := q^{\overline{d}_i} = q^{\frac{(\gamma_i,\gamma_i)^t}{2}},\ \begin{bmatrix} K_i;c\\ p\end{bmatrix} := \prod_{s=1}^{p}\frac{K_i q^{\overline{d}_i(c-s+1)} - K_i^{-1}q^{\overline{d}_i(-c+s-1)}}{q^{\overline{d}_i s} - q^{-\overline{d}_i s}}.$$

Let $U_{q,\mathcal{A}}$ be the form of U_q defined over the ring of Laurent polynomials $\mathcal{A} := \mathbb{Z}[q,q^{-1}]$. We denote by R the ring $\mathcal{A}/I$, where I is the ideal generated by the 2ℓ-th cyclotomic polynomial, and set $U_{q,R} := U_{q,\mathcal{A}} \otimes_{\mathcal{A}} R$.

Similarly, let U_q^+ (respectively U_q^-) be the subalgebra generated by the E_i (F_i), and denote by $U_{q,\mathcal{A}}^+$ (respectively $U_{q,\mathcal{A}}^-$) the subalgebra of $U_{q,\mathcal{A}}$ generated by the divided powers $E_i^{(n)} := \frac{E_i^{(n)}}{[n]_i}$ ($F_i^{(n)} := \frac{F_i^{(n)}}{[n]_i}$). Let $U_{q,R}^+$ be the algebra $U_{q,\mathcal{A}}^+ \otimes_{\mathcal{A}} R$, and denote by $U_{q,R}^-$ the algebra $U_{q,\mathcal{A}}^- \otimes_{\mathcal{A}} R$.

We use a similar notation for the enveloping algebra $U(\mathfrak{g})$. To distinguish between the generators of $U(\mathfrak{g})$ and $U_q(\mathfrak{g}^t)$, we denote the generators of $U(\mathfrak{g})$ by $X_\alpha, H_\alpha, Y_\alpha$ or X_i, H_i, Y_i. Let $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ defined over $\mathbb{Q}$, let $U_\mathbb{Z}$ be the Kostant-$\mathbb{Z}$-form of U, set $U_R := U_\mathbb{Z} \otimes_\mathbb{Z} R$, etc. We fix the symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{H}$ such that $(\alpha_i^\vee, h) = \langle \alpha_i, h\rangle / d_i$, and let $(\cdot,\cdot)^t$ be defined accordingly. We use the forms to identify $\mathfrak{H}$ with $\mathfrak{H}^*$, it will be clear from the context which form we use.

Denote by $\mathbf{v}$ the image of q in R. Set $\ell_i := \frac{\ell d_i}{d}$, then, by the definition of d, ℓ_i is minimal such that

$$\ell_i \frac{(\gamma_i,\gamma_i)^t}{2} = \ell_i \overline{d}_i = \ell_i \frac{d}{d_i} \in \ell\mathbb{Z}.$$

Let $N(\lambda)$ be the simple $U_q(\mathfrak{g}^t)$-module of highest weight $\lambda \in X^{t,+}$, fix an $\mathcal{A}$-lattice $N(\lambda)_\mathcal{A} := U_{q,\mathcal{A}} m_\lambda$ in $N(\lambda)$ by choosing a highest weight vector $m_\lambda \in N(\lambda)$. Set $N(\lambda)_R := N(\lambda)_\mathcal{A} \otimes_\mathcal{A} R$, then $N(\lambda)_R$ is an $U_{q,R}$-module such that its character is given by the Weyl character formula. Consider the weight space decomposition:

$$N(\lambda)_R = \bigoplus_{\mu \in X^t} N(\lambda)_{R,\mu} \quad \text{and set} \quad N(\lambda)_R^{\overline{\ell}} := \bigoplus_{\mu \in \overline{\ell} X} N(\lambda)_{R,\mu}.$$

The subspace $N(\lambda)_R^{\overline{\ell}}$ is stable under the subalgebra of $U_{q,R}$ generated by the $E_i^{(n\ell_i)}$ and $F_i^{(n\ell_i)}$: If $\mu \in \overline{\ell}X$, then so is $\mu \pm n\ell_i\gamma_i = \mu \pm \frac{nd_i\ell}{d}\gamma_i = \mu \pm n\overline{\ell}\alpha_i$.

Theorem 3. *The map*

$$X_i^{(n)} \mapsto E_i^{(n\ell_i)}|_{N(\lambda)_R^{\overline{\ell}}}, \quad Y_i^{(n)} \mapsto F_i^{(n\ell_i)}|_{N(\lambda)_R^{\overline{\ell}}}, \quad \binom{H_i + m}{n} \mapsto \begin{bmatrix} K_i; m\ell_i \\ n\ell_i \end{bmatrix}|_{N(\lambda)_R^{\overline{\ell}}},$$

extends to a representation map $U_R \to \mathrm{End}_R\, N(\lambda)_R^{\overline{\ell}}$.

Some remarks to the proof. One has to prove that the map is compatible with the Serre relations. For U_R^+ and U_R^-, this is a direct consequence of the higher order quantum Serre relations ([23], Chapter 7). For a detailed proof see [23], section 35.2.3. For the proof that also the remaining Serre relations hold see [15].

Let $N = \bigoplus_{\mu \in X^t} N_\mu$ be a finite dimensional $U_q(\mathfrak{g}^t)$-module with a weight space decomposition. If N admits a $U_{q,\mathcal{A}}(\mathfrak{g}^t)$-stable $\mathcal{A}$-lattice $N_\mathcal{A}$ such that $N_\mathcal{A} = \bigoplus_{\mu \in X^t} N_{\mathcal{A},\mu}$ (where $N_{\mathcal{A},\mu} := N_\mathcal{A} \cap N_\mu$), then we denote for any $\mathcal{A}$-algebra R by

N_R the $U_{q,R}(\mathfrak{g}^t)$-module $N_{\mathcal{A}} \otimes_{\mathcal{A}} R$. We have a corresponding weight space decomposition $N_R = \bigoplus_{\mu \in X^t} N_{R,\mu}$.

The same arguments as above show that we can make $N_R^{\overline{\ell}} := \bigoplus_{\mu \in \overline{\ell} X} N_{R,\mu}$ into an $U_R(\mathfrak{g})$-module by the same construction. Let S be the antipode, the action of $U_{q,R}(\mathfrak{g}^t)$ on the dual module $N_R^* := \mathrm{Hom}_R(N_R, R)$ is given by: $(uf)(m) := f(S(u)(m))$ for $u \in U_{q,R}(\mathfrak{g}^t)$ and $f \in N_R^*$. It is easy to check:

Proposition 5. *The map* $U_R \to \mathrm{End}_R \left(N_R^{\overline{\ell}}\right)^*$ *defined by*

$$X_i^{(n)} f(m) := f(S(E_i^{(n\ell_i)})m), \quad Y_i^{(n)} f(m) := f(S(F_i^{(n\ell_i)})m),$$

and $\binom{H_i+k}{n} f(m) := f(S(\left[\begin{smallmatrix} K_i;k\ell_i \\ n\ell_i \end{smallmatrix}\right])m)$, *is the representation map corresponding to the dual representation of the representation of* $U_R(\mathfrak{g})$ *on* $N_R^{\overline{\ell}}$.

4. A basis associated to LS-paths

We use the same notation as before. Let λ be a dominant weight, and let $B(\lambda)$ be the corresponding set of paths obtained from the straight line joining λ and the origin. This set is called the set of LS-paths (Lakshmibai-Seshadri paths) of shape λ. Note if $\pi = \nu_1 * \ldots * \nu_r$ (where $\nu_i \neq \nu_{i+1}$), then we can find $\tau_1, \ldots, \tau_r \in W/W_\lambda$ and $0 < a_1 < a_2 < \ldots < a_r = 1$ such that $\pi = a_1\tau_1(\lambda) * (a_2 - a_1)\tau_2(\lambda) * \ldots * (1 - a_{r-1})\tau_r(\lambda)$. We denote the path π by $(\tau_1, \ldots, \tau_r; 0, a_1, \ldots, a_r = 1)$, and let $i(\pi) := \tau_1$ be the element corresponding to the initial direction of the path.

Denote by $\tilde{R}$ the ring obtained by adjoining all roots of unity to $\mathbb{Z}$. For all $\ell \in \mathbb{N}$ such that $2d$ divides ℓ, fix an embedding $R \hookrightarrow \tilde{R}$.

If k is an algebraically closed field and $\mathrm{char} k = 0$, then we consider k as an R-module by the inclusion $R \hookrightarrow \tilde{R} \subset k$. If $\mathrm{char} k = p > 0$, then we consider k as an R-module by extending the canonical map $\mathbb{Z} \to k$ to a map $\tilde{R} \to k$. For a dominant weight λ denote by $V(\lambda)_{\tilde{R}} = V(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \tilde{R}$ the corresponding Weyl module for $U_{\tilde{R}}(\mathfrak{g})$ over the ring $\tilde{R}$.

By the previous section we know that we have the following sequence of inclusions of $\tilde{R}$-modules, where the top row is an inclusion of $U_{q,\tilde{R}}(\mathfrak{g}^t)$-modules, the bottom row is an inclusion of $U_{\tilde{R}}(\mathfrak{g})$-modules:

$$\begin{array}{ccccc} & & N(\overline{\ell}\lambda)_{\tilde{R}} & \hookrightarrow & \underbrace{N(\lambda)_{\tilde{R}} \otimes \ldots \otimes N(\lambda)_{\tilde{R}}}_{\overline{\ell}} \\ & & \uparrow & & \uparrow \\ V(\lambda)_{\tilde{R}} & \hookrightarrow & \left(N(\overline{\ell}\lambda)_{\tilde{R}}\right)^{\overline{\ell}} & \hookrightarrow & \big(\underbrace{N(\lambda)_{\tilde{R}} \otimes \ldots \otimes N(\lambda)_{\tilde{R}}}_{\overline{\ell}}\big)^{\overline{\ell}} \end{array}$$

The inclusions induce restriction maps for the corresponding dual modules:

$$\begin{array}{ccccc} \big(\underbrace{N(\lambda)_{\tilde{R}} \otimes \ldots \otimes N(\lambda)_{\tilde{R}}}_{\overline{\ell}}\big)^* & \to & N(\overline{\ell}\lambda)_{\tilde{R}}^* & & \\ \downarrow & & \downarrow & & \\ \big((\underbrace{N(\lambda)_{\tilde{R}} \otimes \ldots \otimes N(\lambda)_{\tilde{R}}}_{\overline{\ell}})^{\overline{\ell}}\big)^* & \to & \left((N(\overline{\ell}\lambda)_{\tilde{R}})^{\overline{\ell}}\right)^* & \to & V(\lambda)_{\tilde{R}}^* \end{array}$$

We use these maps to define some special vectors in $V(\lambda)^*_{\tilde{R}}$. Fix a highest weight vector m_λ in $N(\lambda)$ such that $N(\lambda)_{\mathcal{A}} := U_{q,\mathcal{A}}(\mathfrak{g}^t)m_\lambda$ and $N(\lambda)_{\tilde{R}} = N(\lambda)_{\mathcal{A}} \otimes_{\mathcal{A}} \tilde{R}$. We define for $\tau \in W/W(\lambda)$ a *canonical extremal weight vector* m_τ of weight $\tau(\lambda)$ as follows: Fix a reduced decomposition $\tau = s_{i_1} \cdots s_{i_r}$. According to this decomposition let $n_1, \dots, n_r$ be defined by

$$n_r := \langle \alpha^\vee_{i_r}, \lambda \rangle,\ n_{r-1} := \langle \alpha^\vee_{i_{r-1}}, s_{i_r}(\lambda) \rangle, \quad \dots, \quad n_1 := \langle \alpha^\vee_{i_1}, s_{i_2} \cdots s_{i_r}(\lambda) \rangle.$$

We set $m_\tau := F^{(n_1)}_{i_1} \dots F^{(n_r)}_{i_r} m(\lambda)$. The fact that m_τ is independent of the choice of the decomposition follows from the quantum Verma identities.

Denote by $b_\tau \in N(\lambda)^*_{\tilde{R}}$ the corresponding *dual weight vector* of m_τ of weight $-\tau(\lambda)$. We define in the same way extremal weight vectors $v_\tau \in V(\lambda)_{\mathbb{Z}}$ and $p_\tau \in V^*(\lambda)_{\mathbb{Z}}$.

Let $\pi = (\tau_1, \dots, \tau_s; 0, a_1, \dots, 1)$ be an L-S path of shape λ. Suppose ℓ is minimal with the property that $2d$ divides ℓ and $\overline{\ell} a_i \in \mathbb{Z}$ for all $i = 1, \dots, s$. Then we can associate to π the vector

$$b_\pi := \underbrace{b_{\tau_s} \otimes \dots \otimes b_{\tau_s}}_{\overline{\ell}(1-a_{s-1})} \otimes \dots \otimes \underbrace{b_{\tau_2} \otimes \dots \otimes b_{\tau_2}}_{\overline{\ell}(a_2-a_1)} \otimes \underbrace{b_{\tau_1} \otimes \dots \otimes b_{\tau_1}}_{\overline{\ell}a_1} \in (N(\lambda)^*_{\tilde{R}})^{\otimes \overline{\ell}}.$$

Definition 4. We call the image of b_π in $V(\lambda)^*_{\tilde{R}}$ the *path vector* associated to π, and we denote it by p_π. By abuse of notation, we denote by p_π as well its image in $V(\lambda)^*_k = V(\lambda)^*_{\tilde{R}} \otimes_{\tilde{R}} k$ for any algebraically closed field k.

The vector p_π depends only on the path $\pi = (\tau_1, \dots; 0, a_1, \dots, 1)$ (and the choice of $m_\lambda \in N(\lambda)_A$). We use the following partial order on the set of L-S paths $B(\lambda)$ of shape λ: If $\eta = (\kappa_1, \dots; 0, b_1, \dots, 1) \in B(\lambda)$ is another L-S paths, then we write $\pi \geq \eta$ if $\tau_1 > \kappa_1$, or $\tau_1 = \kappa_1$ and $a_1 > b_1$, or $\tau_1 = \kappa_1$, $a_1 = b_1$ and $\tau_2 > \kappa_2$, etc.

The following theorem has been proved in [15]:

Theorem 4. $\mathbb{B}(\lambda) := \{p_\pi \mid \pi \in B(\lambda)\}$ *is a basis of* $V(\lambda)^*_k$ *of* $\mathfrak{H}$*-eigenvectors of weight* $-\pi(1)$*. Further, let* $\pi_1, \pi_2, \dots$ *be a numeration of the L-S paths such that* $\pi_i > \pi_j$ *implies* $i > j$*, and denote by* $V(\lambda)^*_k(j)$ *the subspace spanned by the* p_{π_i}*,* $i \geq j$*. The flag* $\mathbb{V}(\lambda)^*_k$ *is* $U_k(\mathfrak{g})^+$*-stable:*

$$\mathbb{V}(\lambda)^*_k : \ V(\lambda)^*_k = V(\lambda)^*_k(1) \supset V(\lambda)^*_k(2) \supset V(\lambda)^*_k(3) \supset \dots$$

Idea of the proof: We associate to a path $\pi = (\tau_1, \dots; 0, a_1, \dots, 1) \in B(\lambda)$ a vector $v_\pi \in V(\lambda)_{\mathbb{Z}}$ of the form

$$v_\pi := Y^{(n_1)}_{\alpha_{i_1}} \cdots Y^{(n_r)}_{\alpha_{i_r}} v_\lambda,$$

where $v(\lambda)$ is a highest weight vector, $\tau_1 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ is a reduced decomposition, and the sequence $s(\pi) = (n_1, \dots, n_r)$ is given by an algorithm explained below.

We show that the vectors v_π form a basis of the lattice $V(\lambda)_{\mathbb{Z}}$. Further, we will see that $p_\pi(v_\eta) = 0$ for $\eta \not\geq \pi$ and $p_\pi(v_\pi)$ is a root of unity, which implies then that the path vectors p_π form a basis of $V(\lambda)^*_k$ for any algebraically closed field k. Note that the basis given by v_π depends on several non-canonical choices.

The construction of the vectors v_π: We explain now the construction and the properties of the vectors v_π mentioned above. We assume always that $2d$ divides ℓ and $\overline{\ell} := \ell/d$ as before. For $\nu \in X^t$ let $m_\nu \in N(\lambda)_{\tilde{R},\nu}$ be a weight vector. Denote by "$\succ$" the usual partial order on the set of weights. We say $m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}} < m_{\lambda_1} \otimes \ldots \otimes m_{\lambda_{\overline{\ell}}}$ if there exists a j such that $\nu_i = \lambda_i$ for all $i < j$ and $\nu_j \succ \lambda_j$. If π is such that $\overline{\ell}a_i \in \mathbb{Z}$ for all $i = 1, \ldots, s$, then we associate to π the vector

$$\mathfrak{m}^\pi := \underbrace{m_{\tau_1} \otimes \ldots \otimes m_{\tau_1}}_{\overline{\ell}a_1} \otimes \underbrace{m_{\tau_2} \otimes \ldots \otimes m_{\tau_2}}_{\overline{\ell}(a_2-a_1)} \otimes \ldots \otimes \underbrace{m_{\tau_s} \otimes \ldots \otimes m_{\tau_s}}_{\overline{\ell}(1-a_{s-1})} \in (N(\lambda)_{\tilde{R}})^{\otimes \overline{\ell}}.$$

This will be the *leading term* of the vector v_π. Note if $\pi > \eta$ and $\overline{\ell}b_i \in \mathbb{Z}$ for $\eta = (\kappa_1, \ldots, \kappa_t, 0, b_1, \ldots, 1)$, then $\mathfrak{m}^\pi > \mathfrak{m}^\eta$.

The set up for this procedure has been inspired by the article of K. N. Raghavan and P. Sankaran [28]. For $\pi = (\tau_1, \ldots; 0, a_1, \ldots, 1)$ fix a reduced decomposition $\tau_1 = s_{i_1} \ldots s_{i_r}$, the sequence $s(\pi)$ will depend on the chosen reduced decomposition. Fix j minimal such that $s_{i_1}\tau_j > \tau_j$, and set $j = r+1$ if $s_{i_1}\tau_j \leq \tau_j$ for all j. It is easy to see that the path $\eta = (s_{i_1}\tau_1, \ldots, s_{i_1}\tau_{j-1}, \tau_j, \ldots, \tau_r; 0, a_1, \ldots, 1)$ is an L-S path of shape λ (it is understand that we omit a_{j-1} if $s_{i_1}\tau_{j-1} = \tau_j$).

It follows that $\eta(1) - \pi(1)$ is an integral multiple of the simple root α_{i_1}. Let $n_1 \in \mathbb{N}$ be such that $\eta(1) - \pi(1) = n_1\alpha_{i_1}$. Note that $s_{i_1}\tau_1 = s_{i_2} \ldots s_{i_r}$ is a reduced decomposition, and $s_{i_1}\tau_1 < \tau_1$. Suppose we have already defined $s(\eta) = (n_2, \ldots, n_r)$ (where $s(id; 0, 1)$ is the empty sequence). We define the sequence for π to be the one obtained by adding n_1 to the sequence for η:

Definition 5. We denote by $s(\pi)$ the sequence $(n_1, n_2, \ldots, n_r)$, and we associate to π the vector $v_\pi := Y^{(n_1)}_{\alpha_{i_1}} \cdots Y^{(n_r)}_{\alpha_{i_r}} v_\lambda \in V(\lambda)_{\mathbb{Z}}$. We call $i(\pi) := \tau_1$ the *initial direction* of π.

The vector v_π depends on the choice of the reduced decomposition. By construction, we know that v_π is a weight vector of weight $\pi(1)$. The most important step in the proof is the following lemma which describes the image of v_π under the inclusion $V(\lambda)_R \hookrightarrow N(\lambda)^{\otimes\overline{\ell}}_R$. The proof is a simple calculation, for details see [15], Lemma 3.

Lemma 5. *If $\overline{\ell}a_i \in \mathbb{Z}$ for all $i = 1, \ldots, s$, then there exists an $h \in \mathbb{N}$ such that*

$$v_\pi = \mathbf{v}^h \mathfrak{m}^\pi + \sum m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}}$$

where the $m_{\nu_j} \in N(\lambda)_R$ are weight vectors such that $\mathfrak{m}^\pi > m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}}$.

Choose now ℓ big enough such that $\overline{\ell}a_i \in \mathbb{Z}$ for all $\pi \in B(\lambda)$, so for all $\pi \in B(\lambda)$ we have a leading term (Lemma 5) $\mathfrak{m}^\pi \in N(\lambda)^{\otimes\overline{\ell}}_R$ of $v_\pi \in V(\lambda)_R \hookrightarrow N(\lambda)^{\otimes\overline{\ell}}_R$. Since the m_π are obviously linearly independent, the same is true for the v_π. By the Weyl character formula we know hence the the v_π, $\pi \in B(\lambda)$, span an R-lattice in $V(\lambda)_R$ of the desired rank. Note that the set $\{m_\pi \mid \pi \in B(\lambda)\}$ can be extended to an R-basis of $N(\lambda)_R^{\otimes \overline{\ell}}$. It follows easily from this that the v_π form hence an R-basis of $V(\lambda)_R$. But since the v_π are in the Kostant lattice by construction, it follows:

Lemma 6.
a) $\check{\mathbb{B}}(\lambda) := \{v_\pi \mid \pi \in B(\lambda)\}$ *is a basis of* $V(\lambda)_{\mathbb{Z}}$.
b) $X_\alpha^{(n)} v_\pi = \sum_{\pi > \eta} a_\eta v_\eta$.

Proof. It remains to prove part *b*). Suppose again that ℓ is big enough such that $\overline{\ell} a_j \in \mathbb{Z}$ for all paths $\pi \in B(\lambda)$. If we apply a generator $X_\alpha^{(n)}$ to v_π and express the result as a linear combination of tensor products of weight vectors in $N(\lambda)_R$:

$$X_\alpha^{(n)} v_\pi = \sum m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}}$$

then it is easy to see by Lemma 5 that $\mathfrak{m}^\pi > m_{\nu_1} \otimes \ldots \otimes m_{\nu_{\overline{\ell}}}$ for such vectors occurring in the expression. Since the $\mathfrak{m}^\eta$ are linearly independent, in an expression of $X_\alpha^{(n)} v_\pi$ as a linear combination in the v_η:

$$X_\alpha^{(n)} v_\pi = \sum a_\eta v_\eta,$$

the coefficient a_η has to vanish for all $\eta \not< \pi$. $\square$

Proof of Theorem 4. Lemma 5 implies that $p_\pi(v_\eta) = 0$ if $\eta \not\geq \pi$ and $p_\pi(v_\eta)$ is a root of unity. We can hence find an upper triangular matrix, with roots of unities on the diagonal, that transforms the dual basis of $\check{\mathbb{B}}(\lambda)$ into $\mathbb{B}(\lambda)$. It follows that $\mathbb{B}(\lambda)$ is a basis of $V(\lambda)_k^*$ for any algebraically closed field k. The second part of Lemma 6 proves the claim about the $U_k^+(\mathfrak{g})$-stable flags. $\square$

During the rest of this section we will prove a slightly stronger version of Theorem 4. For $\tau \in W/W(\lambda)$ denote by $V(\lambda)_{\mathbb{Z}}(\tau) \subset V(\lambda)_{\mathbb{Z}}$ the submodule $U_{\mathbb{Z}}^+(\mathfrak{g}) v_\tau$. Recall that $V(\lambda)_{\mathbb{Z}}(\tau)$ can also be described as the submodule spanned by all vectors of the form $Y_{\alpha_{i_1}}^{(n_1)} \cdots Y_{\alpha_{i_r}}^{(n_r)} v_\lambda$, where the n_i are arbitrarily chosen and $\tau = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_r}}$ is a reduced decomposition. As an immediate consequence we see: $v_\pi \in V(\lambda)_{\mathbb{Z}}(\tau)$ if $i(\pi) \leq \tau$.

Theorem 5.
a) The set $\{v_\pi \mid \pi \mid \pi \in B(\lambda),\ i(\pi) \leq \tau\}$ *is a basis of* $V(\lambda)_{\mathbb{Z}}(\tau)$.
b) $V(\lambda)_{\mathbb{Z}}(\tau)$ *is a direct summand of* $V(\lambda)_{\mathbb{Z}}$.
c) The restrictions $\{p_\pi|_{V(\lambda)_k(\tau)} \mid \pi \in B(\lambda), i(\pi) \leq \tau\}$*, form a basis of* $V(\lambda)_k^*(\tau)$*, and if* $i(\pi) \not\leq \tau$*, then* $p_\pi|_{V(\lambda)_k(\tau)} \equiv 0$.

Proof. The second and third part is a simple consequence of Lemma 5 and part *a*) of the theorem. Lemma 6 proves that the span of the v_π, $i(\pi) \leq \tau$, is a $U_{\mathbb{Z}}^+(\mathfrak{g})$-stable submodule of $V(\lambda)_{\mathbb{Z}}(\tau)$. Since $v_\tau = v_\eta$ for $\eta = (\tau; 0, 1)$ we know in addition that v_τ is in this submodule, which implies that $V(\lambda)_{\mathbb{Z}}(\tau)$ is equal to the span of the v_π, $i(\pi) \leq \tau$. $\square$

Denote by Λ_α the Demazure operator on the group ring $\mathbb{Z}[X]$:

$$\Lambda_\alpha(e^\mu) := \frac{e^{\mu+\rho} - e^{s_\alpha(\mu+\rho)}}{1 - e^{-\alpha}} e^{-\rho}$$

By the Demazure formula for path models (see [16]) we get:

Corollary 1. $\mathrm{Char}\, V(\lambda)_{\mathbb{Z}}(\tau) = \Lambda_{i_1} \ldots \Lambda_{i_r} e^\lambda$ *for any reduced decomposition* $\tau = s_{i_1} \ldots s_{i_r}$.

5. Schubert varieties

We apply now the results above to the geometry of Schubert varieties. The results obtained above have consequences which seem to be as powerful as the Frobenius splitting introduced in the articles of Mehta, Ramanan and Ramanathan (see for example [25,26,27]). As a consequence of the path basis one gets the normality of Schubert varieties, the vanishing theorems, the reducedness of intersections of unions of Schubert varieties etc. For detailed proofs see [15] and [21]. In fact, the proofs run along the same line as the proofs in [11]. But since the construction of the basis is not anymore part of the inductive procedure, these arguments can be applied in a straight forward way. Note that the cases discussed in [11] are special cases of the situation discussed in this and the preceding section. It is easy to see that for fundamental weights of classical type the basis constructed here and in [11] are the same.

Let k be an algebraically closed field, we will omit the subscript k whenever there is no confusion possible. Let G be the simply connected semisimple group corresponding to $\mathfrak{g}$, and, according to the choice of the triangular decomposition of $\mathfrak{g}$, let $B \subset G$ be a Borel subgroup. Fix a dominant weight λ and let $P \supset B$ be the parabolic subgroup of G associated to λ. It is well-known that the space of global sections $\Gamma(G/P, \mathcal{L}_\lambda)$ of the line bundle $\mathcal{L}_\lambda := G \times_P k(\lambda)$ is, as G-representation, isomorphic to $V(\lambda)^*$. Let $\phi : G/P \hookrightarrow \mathbb{P}(V(\lambda))$ be the corresponding embedding.

For $\tau \in W/W(\lambda)$ denote by $X(\tau) \subset G/P$ the Schubert variety. Let $Y = \bigcup_{i=1}^r X(\tau_i)$ be a union of Schubert varieties. By abuse of notation, we denote by $\mathcal{L}_\lambda$ and p_π also the restrictions $\mathcal{L}_\lambda|_Y$ and $p_\pi|_Y$. Recall that the linear span of the affine cone over $X(\tau)$ in $V(\lambda)$ is the submodule $V(\lambda)(\tau)$. The restriction map $\Gamma(G/P, \mathcal{L}_\lambda) \to \Gamma(X(\tau), \mathcal{L}_\lambda)$ induces hence an injection $V(\lambda)^*(\tau) \hookrightarrow \Gamma(X(\tau), \mathcal{L}_\lambda)$. We call a path vector p_π *standard on* Y if $i(\pi) \leq \tau_i$ for at least one $1 \leq i \leq r$. Denote $\mathbb{B}(\lambda)(Y)$ the set of standard path vectors on Y.

Theorem 6.
a) $\mathbb{B}(\lambda)(Y)$ *is a basis of* $\Gamma(Y, \mathcal{L}_\lambda)$.
b) $p_\pi|_Y \equiv 0$ *if and only if* $i(\pi) \not\leq \tau_i$ *for all* $i = 1, \ldots, r$.

As an immediate consequence we get:

Corollary.

a) The restriction map $\Gamma(G/P, \mathcal{L}_\lambda) \to \Gamma(Y, \mathcal{L}_\lambda)$ *is surjective.*

b) For any reduced decomposition $\tau = s_{i_1} \dots s_{i_r}$, $\operatorname{Char} \Gamma(X(\tau), \mathcal{L}_\lambda)^*$ *is given by the Demazure character formula* $\operatorname{Char} \Gamma(X(\tau), \mathcal{L}_\lambda)^* = \Lambda_{i_1} \dots \Lambda_{i_r} e^\lambda$.

The proof of the theorem is by induction on the dimension and the number of irreducible components of maximal dimension. Let Y, Y_1, Y_2 be unions of Schubert varieties. During the induction procedure one proves in addition:

Theorem 7.

i) $H^i(Y, \mathcal{L}_\lambda) = 0$ *for* $i \geq 1$.

ii) $X(\tau)$ *is a normal variety.*

iii) The scheme theoretic intersection $Y_1 \cap Y_2$ *is reduced.*

6. Standard Monomials

Let $\lambda_1, \dots, \lambda_r$ be some dominant weights, set $\lambda = \sum \lambda_i$, and fix $\tau \in W/W(\lambda)$. For each i let τ_i be the image of τ in W/W_{λ_i}. A module $V(\lambda)$ (without specifying the underlying ring) is always meant to be a module over an algebraically closed field. The inclusion $V(\lambda) \hookrightarrow V(\lambda_1) \otimes \dots \otimes V(\lambda_r)$ induces an inclusion $V_\tau(\lambda) \hookrightarrow V(\lambda_1)(\tau_1) \otimes \dots \otimes V(\lambda_r)(\tau_r)$, and hence in turn the restriction map $V(\lambda_1)^*(\tau_1) \otimes \dots \otimes V(\lambda_r)^*(\tau_r) \to V(\lambda)^*(\tau)$.

We write π_i respectively π_λ for the path $t \mapsto t\lambda_i$ respectively $t \mapsto t\lambda$. Denote by $\mathbb{B}_i$ respectively $\mathbb{B}(\lambda)$ the set of L-S paths of shape λ_i respectively λ.

Denote by $\mathbb{B}_1 * \dots * \mathbb{B}_r$ the set of concatenations of all paths in $\mathbb{B}_1, \dots, \mathbb{B}_r$. The set of paths $\mathbb{B}_1 * \dots * \mathbb{B}_r \cup \{0\}$is stable under the root operators. Denote by $B(\pi_1 * \dots * \pi_r)$ the smallest subset such that $B(\pi_1 * \dots * \pi_r) \cup \{0\}$ is stable under the root operators and contains $\pi_1 * \dots * \pi_r$.

A monomial $\eta_1 * \dots * \eta_r \in \mathbb{B}_1 * \dots * \mathbb{B}_r$ is called *standard* if it an element of $B(\pi_1 * \dots * \pi_r)$.

A more combinatorial way of defining $i(\eta_1 * \dots * \eta_r)$ is via the *defining chain*. Let first $\eta := \eta_1 * \dots * \eta_r$ be an arbitrary element of $\mathbb{B}_1 * \dots * \mathbb{B}_r$, where $\eta_i = (\tau_1^i, \dots, \tau_{s_i}^i; 0, a_1, \dots, 1)$. If the λ_i belong to different faces of the dominant Weyl chamber, then there is no way of comparing the τ_j^i using the Bruhat order.

Choose lifts $w_j^i \equiv \tau_j^i \mod W_{\lambda_i}$ with respect to the projection maps $W/W_\lambda \to W/W_{\lambda_i}$ for $1 \leq i \leq r, 1 \leq j \leq s_i$. If one can choose the w_j^i such that we have for $\underline{w} := (w_1^1, \dots, w_{s_r}^r)$:

$$w_1^1 \geq w_2^1 \geq \dots \geq w_{s_1}^1 \geq w_1^2 \geq w_2^2 \geq \dots \geq w_1^r \geq w_2^r \geq \dots \geq w_{s_r}^r.$$

then we call $\underline{w}$ a *defining chain for* η.

Such a defining chain exists if and only if η is standard, i.e. $\eta \in \mathbb{B}_1 * \dots * \mathbb{B}_r$ [18]. Note that if $\lambda = \lambda_i$ for all i, then we get in this way a very simple criteria for η to be standard: η is standard iff

$$\tau_1^1 \geq \tau_2^1 \geq \dots \geq \tau_{s_1}^1 \geq \tau_1^2 \geq \tau_2^2 \geq \dots \geq \tau_1^r \geq \tau_2^r \geq \dots \geq \tau_{s_r}^r.$$

The defining chain (if it exists) is in general not necessarily unique. We introduce a partial ordering on these chains by saying $\underline{w} \geq \underline{w}'$ if $w^i_j \geq {w'}^i_j$ for all i, j. There exists a unique minimal and a unique maximal defining chain. If η is standard, then we set $i(\eta) := w^1_1$ for the minimal defining chain of η. See [18] for a proof that the two notions are the same.

Definition 6. Let $\eta_1, \ldots, \eta_r$ be L-S paths of shape $\lambda_1, \ldots, \lambda_r$. A monomial of path vectors $p_{\eta_1} \cdots p_{\eta_r}$ is called *standard* if the concatenation $\eta_1 * \ldots * \eta_r$ is standard. The standard monomial is called *standard with respect to* τ if $i(\eta_1 * \ldots * \eta_r) \leq \tau$.

For a detailed proof we refer to [15] and [21].

Theorem 8. *The set of standard monomials form a basis of* $V(\lambda)^*$*, and the set of monomials standard with respect to* τ *form a basis for* $V(\lambda)^*(\tau)$.

Remark 3. We have seen that the p_η, η not standard with respect to τ, form a basis of the kernel of the restriction map $V_k(\lambda)^* \to V_k(\lambda)^*(\tau)$. This not true in general for standard monomials if the λ_i have different stabilizers. The reason for this is that the partial order on the tensor products of weight vectors is not anymore related to $i(\eta_1 * \ldots * \eta_r)$. Let $\mathfrak{H}$ be the Cartan subalgebra of diagonal matrices of trace zero of $\mathfrak{g} = \mathfrak{sl}_4$, let ϵ_i be the character that projects a diagonal matrix onto its i-th entry, and let $\omega_i = \epsilon_1 + \ldots + \epsilon_i$ be the i-th fundamental weight. Set $\lambda_1 := \omega_1$, $\eta_1 := (s_2 s_1; 0, 1)$, $\pi_1 := (s_1; 0, 1)$, and set $\lambda_2 := \omega_3$, $\eta_2 := (s_3; 0, 1)$, $\pi_2 := (s_2 s_3; 0, 1)$. The concatenations $\eta_1 * \eta_2$ and $\pi_1 * \pi_2$ are standard, $\eta_1 * \eta_2 > \pi_1 * \pi_2$, but $i(\eta_1 * \eta_2) = s_2 s_1 s_3$, whereas $i(\pi_1 * \pi_2) = s_1 s_2 s_3$. These two are not compatible. It is easy to check that $p_{\pi_1} p_{\pi_2}(v_{\eta_1 * \eta_2}) \neq 0$, so the restriction of $p_{\pi_1} p_{\pi_2}$ to $V_k(\omega_1 + \omega_3)(s_1 s_2 s_3)$ does not vanish though $i(\pi_1 * \pi_2) \not\leq s_1 s_2 s_3$.

Let Y be a union of Schubert varieties in G/Q and let $\mathcal{L}_\lambda, \mathcal{L}_\mu$ be base point free line bundles on G/Q. As an immediate consequence of the Basis Theorem for standard monomials we get:

Corollary.
i) The product map $\Gamma(Y, \mathcal{L}_\lambda) \otimes \Gamma(Y, \mathcal{L}_\mu) \to \Gamma(Y, \mathcal{L}_{\lambda+\mu})$ *is surjective.*
ii) The product map $S^n\Gamma(Y, \mathcal{L}_\lambda) \to \Gamma(Y, \mathcal{L}_{n\lambda})$ *is surjective.*

7. Good filtrations I

The following combinatorial construction is part of a work in progress together P. Magyar and V. Lakshmibai [10]. We fix in the following $\lambda, \mu \in X^+$ and $\tau \in W/W_\lambda$. Let λ, μ also denote the paths that join the origin with λ respectively μ by a straight line, and let $B(\lambda)_\tau$ be the set of L–S paths $\eta = (\tau_1, \ldots, \tau_r; 0, \ldots, 1)$ such that $\tau_1 \leq \tau$. Recall that $\operatorname{Char} H^0(X(\tau), \mathcal{L}_\lambda)$ is given by $\sum_{\eta \in B(\lambda)_\tau} e^{\eta(1)}$, and the path vectors $p_\eta|_{X(\tau)}$, $\eta \in B(\lambda)_\tau$, form a basis of $H^0(X(\tau), \mathcal{L}_\lambda)$.

The decomposition formula (see section 1) implies that

$$B(\lambda) * B(\mu) = B(\lambda * \eta_1) \cup \ldots \cup B(\lambda * \eta_s),$$

where $\eta_1, \ldots, \eta_s$ runs through the set of all paths in $B(\mu)$ such that $\operatorname{Im}(\lambda * \eta) \subset \overline{C_0}$. Since the paths are affine, this is equivalent to say that $\lambda * \eta_1, \ldots, \lambda * \eta_s$ runs through the set of all paths in $B(\lambda) * B(\mu)$ such that $e_\alpha(\lambda * \eta) = 0$ for all simple roots.

The formula for the action of the root operators on a concatenation of paths implies that $\lambda * B(\mu)_\tau \subset B(\lambda) * B(\mu)$ is a subset stable under the e_α. Denote by $D(\mu, i)$ or $D(\mu, \eta_i)$ the subset:

$$D(\mu, \eta_i) := \{\eta \in B(\mu)_\tau \mid \lambda * \eta \in B(\lambda * \eta_i)\}.$$

By section 1, this definition is equivalent to

$$D(\mu, \eta_i) := \{\eta \in B(\mu)_\tau \mid \exists\, i_1, i_2, i_3, \dots \text{ such that } e^{n_1}_{\alpha_{i_1}} e^{n_2}_{\alpha_{i_2}} e^{n_3}_{\alpha_{i_3}} \dots (\lambda * \eta) = \lambda * \eta_i\}$$

Since the union of the $B(\lambda * \eta_i)$ is disjoint we have obviously:

Lemma 7. $\lambda * B(\mu)_\tau = \bigcup_{i=1}^{s} \lambda * D(\mu, i)$

To describe the character of such a subset we attach to each $D(\mu, k)$, $1 \leq k \leq s$, a dominant weight ν_k and a union of Schubert varieties Y_k such that $\sum_{\eta \in D(\mu,k)} e^{\lambda + \eta(1)}$ is the character $\operatorname{Char} H^0(Y_k, \mathcal{L}_{\nu_k})$.

The obvious candidat for the dominant weight is:

Definition 7. $\nu_k := \lambda + \eta_k(1)$

The definition of Y_k is more involved. Let $F_1, \dots, F_r$ be the open proper subfaces of the dominant Weyl chamber ($\neq$ interior of the chamber) met by $\lambda + \eta_k([0, 1])$, and counted with multiplicity. I. e., there exist $0 \leq i_1 \leq j_1 \leq \dots \leq i_r \leq j_r \leq 1$ such that η_k is linear on $[i_\ell, j_\ell]$, $\lambda + \eta_k(]i_\ell, j_\ell[) \subset F_\ell$ if $i_\ell < j_\ell$ respectively $\lambda + \eta_k(i_\ell) \in F_\ell$ if $i_\ell = j_\ell$, and $\lambda + \eta_k(t)$ is an element of the interior of the dominant Weyl chamber for $t \notin \bigcup_{\ell=1}^{r} [i_\ell, j_\ell]$.

Since $\lambda + \eta_k([i_\ell, j_\ell]) \subset \overline{F}_\ell$, if $i_\ell < j_\ell$ and $\lambda + \eta_k(i_\ell) \notin F_\ell$, then we may assume that $i_{\ell-1} = j_{\ell-1} = i_\ell$, and if $\lambda + \eta_k(j_\ell) \notin F_\ell$, then we may assume that $i_{\ell+1} = j_{\ell+1} = j_\ell$.

Similarly, if $i_\ell = j_\ell = i_{\ell+1}$, then we may assume that $i_{\ell+1} < j_{\ell+1}$, and if $j_{\ell-1} = i_\ell = j_\ell$, then we may assume that $i_{\ell-1} < j_{\ell-1}$.

Denote by W_ℓ the stabilizer in W of the face F_ℓ. We set $\chi_{r+1} := \nu_k$, and we define inductively χ_ℓ, $1 \leq \ell \leq r$, to be the unique weight in $W_\ell(\chi_{\ell+1})$ such that $\langle \chi_\ell, \alpha^\vee \rangle \leq 0$ for all simple roots α such that $s_\alpha(F_\ell) = F_\ell$.

We associate now to the triple $(\lambda, D(\mu, \eta_k), \nu_k)$ a union of Schubert varieties. Let $Q_k \supset B$ be the parabolic subgroup such that its Weyl group W_{Q_k} is equal to W_{ν_k}, and suppose $\eta_k = (\tau_1, \dots; 0, a_1, \dots)$.

Definition 8. If λ is a regular dominant weight, then let $\sigma_k \in W/W_{\nu_k}$ be the unique element such that $\sigma_k(\nu_k) = \chi_1$ and set $Y_k := X(\sigma_k) \subset G/Q_k$.

If λ is a singular weight (and hence $\lambda \in F_1$), then let $\sigma_k \in W/W_{\nu_k}$ be the unique element such that $\sigma_k(\nu_k) = \chi_2$ and set $Y_k := \bigcup X(s_{i_1} \cdots s_{i_t} \sigma_k) \subset G/Q_k$, where the union runs over all sequences $i_1, \dots, i_t$ such that $s_{i_j}(F_1) = F_1$ for all $j = 1, \dots, t$ and $\tau_1 \leq s_{i_t}\tau_1 \leq \dots \leq s_{i_1} \cdots s_{i_t}\tau_1 \leq \tau$.

Let Φ_ℓ be the set of simple roots γ such that $s_\gamma(F_\ell) = F_\ell$.

Theorem 9.
a) Char $H^0(Y_k, \mathcal{L}_{\nu_k}) = \sum_{\eta\in D(\mu,k)} e^{\lambda+\eta(1)}$
b) $D(\mu, k) =$

$$\big\{\eta \in B(\mu)_\tau \mid \pi_\lambda * \eta = \underbrace{f^{n_1}_{\gamma_{i_1}} \cdots f^{n_{p_1}}_{\gamma_{i_{p_1}}}}_{\gamma\in\Phi_1} \underbrace{f^{m_1}_{\gamma_{j_1}} \cdots f^{n_{p_2}}_{\gamma_{j_{p_2}}}}_{\gamma\in\Phi_2} \cdots \underbrace{f^{q_1}_{\gamma_{\ell_1}} \cdots f^{q_{p_r}}_{\gamma_{\ell_{p_r}}}}_{\gamma\in\Phi_r} (\pi_\lambda * \eta_k)\big\}$$

where the $\gamma \in \Phi_1$ run over all sequences such that $\tau_1 \le s_{\gamma_{i_{p_1}}}\tau_1 \le \ldots \le s_{\gamma_{i_1}} \cdots s_{\gamma_{i_{p_1}}}\tau_1 \le \tau$ in W/W_{ν_k}.

Proof. We show first that the second part of the theorem implies part *a*). Let $\nu \in X^+$ be an arbitrary dominant weight. In [16] we have shown that if $s_\alpha\tau > \tau$, then

$$B(\nu)_{s_\alpha\tau} \cup \{0\} = \bigcup_{n\ge 0} \big\{f^n_\alpha\eta \mid \eta \in B(\nu)_\tau\big\},$$

and $B(\nu)_\tau \cup \{0\}$ is stable under the operator f_α if $s_\alpha\tau \le \tau$. Let now π be a path such that $\pi(1) = \nu$ and $\operatorname{Im}\pi$ is contained in the dominant Weyl chamber $\overline{C}_0$, and define $B(\pi)_\tau$ to be the subset

$$B(\pi)_\tau := \big\{\eta \in B(\pi) \mid \eta = f^{n_1}_{\alpha_{i_1}} \cdots f^{n_q}_{\alpha_{i_q}}(\pi)\big\},$$

where $\tau = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_q}}$ is a reduced decomposition. The graph isomorphism (Theorem 2) implies that $\operatorname{Char} B(\pi)_\tau = \operatorname{Char} B(\nu)_\tau$, $B(\pi)_\tau \cup \{0\}$ is stable under the operator f_α if $s_\alpha\tau \le \tau$, and $B(\pi)_{s_\alpha\tau} \cup \{0\} = \bigcup_{n\ge0}\{f^n_\alpha\eta \mid \eta \in B(\pi)_\tau\}$ if $s_\alpha\tau > \tau$. It follows that the sum $\sum e^{\lambda+\eta(1)}$ running over the elements on the right hand side of *b*) is equal to $\operatorname{Char} H^0(Y_k, \mathcal{L}_{\nu_k})$, so *b*) implies *a*).

It remains to prove *b*). Since $(\lambda + \eta_k([0, 1])) \cap F_r \ne \emptyset$, it follows by Lemma 2 that if $\alpha \in \Phi_r$, then $f_\alpha(\pi_\lambda * \eta_k) = \pi_\lambda * (f_\alpha\eta_k)$, and η_k differs from $f_\alpha\eta_k$ only for $t \ge j_r$. The same argument implies that

$$\underbrace{f^{m_1}_{\gamma_{j_1}} \cdots f^{n_{p_2}}_{\gamma_{j_{p_2}}}}_{\gamma\in\Phi_2} \cdots \underbrace{f^{q_1}_{\gamma_{\ell_1}} \cdots f^{q_{p_r}}_{\gamma_{\ell_{p_r}}}}_{\gamma\in\Phi_r}(\pi_\lambda * \eta_k) = \pi_\lambda * (f^{m_1}_{\gamma_{j_1}} \cdots f^{q_{p_r}}_{\gamma_{\ell_{p_r}}}\eta_k),$$

and the latter differs from η_k only for $t \ge j_2 > 0$. In particular, these paths are of the form $\eta' = (\tau_1, \tau_2', \ldots)$, where the τ_1 is the same as the "first direction" for the path η_k, so $\eta' \in B(\mu)_\tau$.

If λ is a regular weight, then $j_1 > 0$, so the same arguments as above apply to prove that the paths obtained on the right hand side of *b*) are of the form $\eta'' = (\tau_1, \tau_2'', \ldots)$, in particular, $\eta'' \in B(\mu)_\tau$.

If λ is a singular weight, then the condition on the sequences $\gamma_{i_1}, \ldots, \gamma_{i_{p_1}} \in \Phi_1$ imply that the paths on the right hand side of *b*) are of the form $\eta'' = (\tau_1'', \tau_2'', \ldots)$, where $\tau_1'' \le \tau$, so $\eta'' \in B(\mu)_\tau$.

It follows that the right hand side of *b*) is a subset of $D(\mu, k)$. It remains to prove that the two sets are equal. Fix a path $\eta \in D(\mu, k)$. Let t_0 be maximal such that

$\lambda+\eta(t_0)\in\overline{C}_0$. If $t_0=1$, then $\eta=\eta_k$ and nothing is to prove. Otherwise, let F be the face such that $\lambda+\eta(t_0)\in F$. By the maximality of t_0, there exists a simple root α such that $s_\alpha(F)=F$ and $\langle\lambda+\eta(t),\alpha^\vee\rangle<0$ for $t\in]t_0,t_0+\epsilon]$ for some $1\gg\epsilon>0$. It follows by Lemma 1 and Lemma 3 that $e_\alpha(\lambda*\eta)\neq 0$. Let m be maximal such that $e_\alpha^m(\lambda*\eta)\neq 0$, by the choice of t_0 we know that $e_\alpha^m(\lambda*\eta)=\lambda*(e_\alpha^m\eta)$, and $e_\alpha^m\eta(t)=\eta(t)$ for $0\leq t\leq t_0$. By repeating the procedure with other simple roots α_{i_j} such that $s_{i_j}(F)=F$, we get a path $\eta'=e_{\alpha_{i_t}}^{m_t}\cdots e_{\alpha_{i_1}}^{m_1}\eta$ such that:

$$f_{\alpha_{i_1}}^{m_1}\cdots f_{\alpha_{i_t}}^{m_t}(\lambda*\eta')=\lambda*\eta,\quad \eta(t)=\eta'(t)\text{ for }0\leq t\leq t_0,\quad s_{i_j}(F)=F,\ 1\leq j\leq t,$$

and $\langle\lambda+\eta(t),\alpha^\vee\rangle\geq 0$ for $t\in[0,t_0+\epsilon]$ for some $1\gg\epsilon>0$. So $\lambda+\eta'$ stays "longer" in $\overline{C}_0$ then $\lambda+\eta$. Note that the path η' still meets the face F.

We proceed now in the same way with η': Let t_0' be maximal such that $\lambda+\eta(t_0')\in\overline{C}_0$. If $t_0'=1$, then $\eta'=\eta_k$ and η is hence an element of the set defined on the right hand side of *b*). Otherwise, let F' be the face such that $\lambda+\eta'(t_0')\in F'$. By the maximality of t_0', there exists a simple root α such that $s_\alpha(F')=F'$ and $\langle\lambda+\eta'(t),\alpha^\vee\rangle<0$ for $t\in]t_0',t_0'+\epsilon]$ for some $1\gg\epsilon>0$. It follows as above that $e_\alpha(\lambda*\eta')\neq 0$.

We can find simple roots γ_{j_k} such that $s_{j_k}(F')=F'$ and $\eta''=e_{\gamma_{j_s}}^{n_s}\cdots e_{\gamma_{j_1}}^{n_1}\eta'$ is such that:

$$f_{\alpha_{i_1}}^{m_1}\cdots f_{\alpha_{i_t}}^{m_t}\big(f_{\gamma_{j_1}}^{n_1}\cdots f_{\gamma_{j_s}}^{n_s}(\lambda*\eta'')\big)=f_{\alpha_{i_1}}^{m_1}\cdots f_{\alpha_{i_t}}^{m_t}(\lambda*\eta')=\lambda*\eta,$$

and $\langle\lambda+\eta''(t),\gamma^\vee\rangle\geq 0$ for $t\in[0,t_0'+\epsilon]$ for some $1\gg\epsilon>0$. Note that $\eta''(t_0)\in F$ and $s_{\alpha_{i_j}}(F)=F$, and $\eta''(t_0')\in F'$ and $s_{\gamma_{j_k}}(F')=F'$.

Proceeding in the same way with η'', we see that after a finite number of steps we get a path $\tilde\eta$ such that $\operatorname{Im}\lambda*\tilde\eta$ is contained in $\overline{C}_0$ chamber, so $\tilde\eta$ is equal to η_k for some $1\leq k\leq s$. Further, we get a sequence $t_0<t_1<\ldots<1$ such that $\eta_k(t_0)\in F$, $\eta_k(t_0')\in F'$, …, and sequences of simple roots $\alpha_{i_1},\ldots,\alpha_{i_t},\gamma_{j_1},\ldots,\gamma_{j_s},\ldots$, such that $s_{\alpha_{i_j}}(F)=F$, $s_{\gamma_{j_k}}(F')=F'$, …, and

$$f_{\alpha_{i_1}}^{m_1}\cdots f_{\alpha_{i_t}}^{m_t}\big(f_{\gamma_{j_1}}^{n_1}\cdots f_{\gamma_{j_s}}^{n_s}\big(\ldots(\lambda*\eta_k)\ldots\big)\big)=\lambda*\eta. \tag{1}$$

This proves part *b*) of the theorem for the case where λ is a regular dominant weight. If λ is a singular weight, then we have two cases to consider: Suppose first $t_0\neq 0$. In this case $F=F_i$ (in the definition of Y_k) for some $i\geq 2$, so (1) satisfies automatically the condition on the sequence of simple roots from Φ_1 because the sequence is empty in this case.

Next suppose $t_0=0$ and hence $\lambda\in F$ and $F=F_1$. If $\eta=(\sigma_1,\ldots)$, then $\tau\geq\sigma_1$ because $\eta\in B(\mu)_\tau$. By the very definition of the sequence $\alpha_{i_1},\ldots,\alpha_{i_t}\in\Phi_1$ we know that

$$\tau\geq\sigma_1>s_{\alpha_{i_t}}\sigma_1>s_{\alpha_{i_{t-1}}}s_{\alpha_{i_t}}\sigma_1>\ldots>\tau_1=s_{\alpha_{i_1}}\cdots s_{\alpha_{i_t}}\sigma_1,$$

where $\eta_k=(\tau_1,\ldots)$ (see [16], the action of the root operators on L-S paths). It follows that this sequence satisfies the condition in *b*), so we have proved part *b*) of the theorem also in this case. $\square$

8. Good filtrations II

This section is a generalization of the results proved in [19]. In characteristic 0 we know that the tensor product of two irreducible modules decomposes into the direct sum of irreducible modules, and the multiplicities are described by the tensor product rule given in section 1. In positive characteristic one knows [24] that the module $H^0(G/B,\mathcal{L}_\lambda)\otimes H^0(G/B,\mathcal{L}_\mu)$ admits a G-stable filtration such that all subquotients are again isomorphic to $H^0(G/B,\mathcal{L}_\nu)$ for some dominant weight ν. We will give a proof of this "good filtration"-property using the path bases. The idea of the proof is very similar to that in [19], but because of the more general results on B-stable flags (Theorem 4), the proof is much less technical then the one in [19].

We use the same notation as in the preceding sections. We consider the representation, quantum groups and enveloping algebras over an algebraically closed field k. We fix in the following a dominant weight $\lambda\in X^+$ and a total order on W/W_λ which refines the Bruhat order. We denote by $\geq_t$ the corresponding induced total order on the set of L-S paths $B(\lambda)$ (see section 4). Let $\pi_1,\pi_2,\dots$ be a numeration of the L-S paths such that $\pi_i\geq_t\pi_j$ if and only if $i\geq j$. By Theorem 4 we know that the flag

$$\mathbb{V}(\lambda)^*:\ V(\lambda)^*=V(\lambda)^*(1)\supset V(\lambda)^*(2)\supset V(\lambda)^*(3)\supset\dots$$

is $U(\mathfrak{g})^+$-stable, where $V(\lambda)^*(j)$ denotes the subspace spanned by the p_{π_i}, $i\geq j$.

Let $\mu\in X^+$ be a dominant weight, and let $\pi_{i_1},\dots,\pi_{i_s}\in B(\lambda)$ be the paths such that $\mu+\mathrm{Im}(\pi_{i_j})$ is completely contained in the dominant Weyl chamber.

Denote by $\mathcal{F}_r$ the subspace of $V(\lambda)^*$ spanned by all p_{π_j}, $j\geq i_r$, and by $\mathcal{U}_r$ the subspace spanned by all p_{π_j}, $j>i_r$. If $\mathcal{U}_r\neq\mathcal{F}_{r+1}$, then denote by $\widetilde{\mathcal{U}_r/\mathcal{F}_{r+1}}$ the vector bundle $G\times^B\mathcal{U}_r/\mathcal{F}_{r+1}$ on G/B associated to the B-module $\mathcal{U}_r/\mathcal{F}_{r+1}$.

Theorem 10. *If $\mathcal{U}_r\neq\mathcal{F}_{r+1}$, then $H^i(G/B,\mathcal{L}_\mu\otimes\widetilde{\mathcal{U}_r/\mathcal{F}_{r+1}})=0$ for all $i\geq 0$.*

Corollary. *The tensor product $H^0(G/B,\mathcal{L}_\mu)\otimes H^0(G/B,\mathcal{L}_\lambda)$ admits a G-stable filtration*

$$W_1=H^0(G/B,\mathcal{L}_\mu)\otimes H^0(G/B,\mathcal{L}_\lambda)\supset W_2\supset\dots\supset W_s\supset W_{s+1}=0$$

such that the subquotient W_j/W_{j+1} is isomorphic to $H^0(G/B,\mathcal{L}_{\mu+\pi_{i_j}(1)})$ for all $j=1,\dots,s$.

Proof of the corollary. The vanishing of the higher cohomology groups implies that we have an isomorphism

$$H^0(G/B,\mathcal{L}_\mu)\otimes H^0(G/B,\mathcal{L}_\lambda)\simeq H^0(G/B,\mathcal{L}_\mu\otimes H^0(\widetilde{G/B},\mathcal{L}_\lambda)),$$

and $H^i(G/B,\mathcal{L}_\mu\otimes H^0(\widetilde{G/B},\mathcal{L}_\lambda))=0$ for $i\geq 1$. The theorem above implies that the B-module $k_{-\mu}\otimes H^0(G/B,\mathcal{L}_\lambda)$ admits a filtration such that the subquotient is either a one-dimensional B-module corresponding to the weight $-\mu-\pi_{i_j}(1)$, so the corresponding line bundle has as cohomology groups $H^0(G/B,\mathcal{L}_{\mu+\pi_{i_j}(1)})$ and $H^i(G/B,\mathcal{L}_{\mu+\pi_{i_j}(1)})=0$ for $i\geq 1$, or the subquotient is a module such that all cohomology groups of the associated vector bundle on G/B vanish.

It follows that the filtration of the B-module $k_{-\mu}\otimes H^0(G/B,\mathcal{L}_\lambda)$ induces a G-stable filtration $W_1\supset\ldots\supset W_s\supset W_{s+1}=0$ such that the subquotients are isomorphic to $H^0(G/B,\mathcal{L}_{\mu+\pi_{i_j}(1)})$ for all $j=1,\ldots,s$. $\square$

Proof of the theorem. Suppose r is such that $\mathcal{F}_{r+1}\neq\mathcal{U}_r$, and let $\eta\in B(\lambda)$ be such that $p_\eta\in\mathcal{U}_r$ but $p_\eta\notin\mathcal{F}_{r+1}$. It follows that $\mu+\operatorname{Im}\eta$ is not completely contained in the dominant Weyl chamber. Fix t_0 (>0) minimal such that $(\mu+\eta(t_0),\alpha^\vee)=-1$ for some simple root α. Denote by $\Omega(\eta)$ the set of paths $\eta'\in B(\lambda)$ such that $\eta'(t)=\eta(t)$ for all $0\leq t\leq t_0$. By the choice of the total order we know that $p_{\eta'}\in\mathcal{U}_r$ but $\notin\mathcal{F}_{r+1}$ for all $\eta'\in\Omega(\eta)$. Note that if n is maximal such that $f_\alpha^n(\mu*\eta')\neq 0$, then $f_\alpha^j\eta'\in\Omega(\eta)$ for $j=0,\ldots,n$, and if m is maximal such that $e_\alpha^m(\mu*\eta')\neq 0$, then $e_\alpha^j\eta'\in\Omega(\eta)$ for $j=0,\ldots,m-1$.

Fix $\eta_1,\ldots,\eta_p$ such that $\{\pi\in B(\lambda)\mid p_\pi\in\mathcal{U}_r,p_\pi\notin\mathcal{F}_{r+1}\}$ is the disjoint union of the $\Omega(\eta_i)$. We may assume that the numeration is such that $\eta_p\geq_t\ldots\geq_t\eta_1$. We refine the filtration $\mathcal{U}_r\supset\mathcal{F}_{r+1}$ by defining $\mathcal{U}_{r,j}$ as the span of all p_π such that $\pi\geq\eta_j$ or $\pi\in\Omega(\eta_j)$. We get a B-stable filtration

$$\mathcal{U}_r=\mathcal{U}_{r,1}\supset\mathcal{U}_{r,2}\supset\ldots\mathcal{U}_{r,p}\supset\mathcal{U}_{r,p+1}:=\mathcal{F}_{r+1}.$$

To prove the theorem, it suffices to prove that $H^i(G/B,\mathcal{L}_\mu\otimes\widetilde{\mathcal{U}_{r,j}/\mathcal{U}_{r,j+1}})=0$ for all $i\geq 0$ and $j=1,\ldots,p$. Note that the images of the p_η, $\eta\in\Omega(\eta_j)$, form a basis of $\mathcal{U}_{r,j}/\mathcal{U}_{r,j+1}$.

Let $U_{r,j}$ be the span of all vectors v_η such that $\eta\leq\eta_j$ or $\eta\in\Omega(\eta_j)$, and let $U_{r,0}$ be the span of all vectors v_η such that $\eta\leq\pi_{i_r}$. The restriction map $V_\lambda^*\to U_{r,j}^*$ induces an isomorphism of B-modules $\mathcal{U}_{r,j}/\mathcal{U}_{r,j+1}\to(U_{r,j}/U_{r,j-1})^*$.

Let $SL_2(\alpha)\subset G$ be the subgroup corresponding to the simple root α, and denote by T_α its maximal torus contained in T and by B_α its Borel subgroup contained in B. Denote by $P(\alpha)=SL_2(\alpha)B$ the associated minimal parabolic subgroup.

We show in Lemma 8 (see below) that $U_{r,j}/U_{r,j-1}$ admits a B-stable filtration such that the unipotent radical of $P(\alpha)$ acts trivially on the subquotients, and the B_α-module structure on the subquotients comes from an $SL_2(\alpha)$-structure, twisted by the T_α-character $\eta_j(t_0)$. Note that this finishes the proof: $U_{r,j}/U_{r,j-1}$ admits a filtration such that the subquotients are isomorphic to $k_{\eta(t_0)}\otimes W$ for some $SL_2(\alpha)$-modules W, so $\mathcal{U}_{r,j}/\mathcal{U}_{r,j+1}$ admits a filtration such that the subquotients are isomorphic to $k_{-\eta(t_0)}\otimes W^*$, which implies for the associated vector bundle:

$$H^i(SL_2(\alpha)/B_\alpha,\mathcal{L}_{\mu+\eta(t_0)}\otimes\widetilde{W^*})\simeq H^i(SL_2(\alpha)/B_\alpha,\mathcal{L}_{\mu+\eta(t_0)})\otimes W^*=0\quad\forall\, i\geq 0,$$

because $(\mu+\eta(t_0),\alpha^\vee)=-1$. Using the filtration and the Leray spectral sequence associated to the projection $G/B\to G/P(\alpha)$, it follows that $H^i(G/B,\mathcal{L}_{\mu+\eta(t_0)}\otimes\widetilde{\mathcal{U}_{r,j}/\mathcal{U}_{r,j+1}})=0$ for all $i\geq 0$. $\square$

It remains to define the filtration and the $U(\mathfrak{sl}_2(\alpha))$-structure. We consider the following more general situation: Let $\eta = (\tau_1, \dots, \tau_r; 0, a_1, \dots, 1)$ be an L-S path of shape λ and fix a simple root α. Suppose $t_0 \in [0,1]$ is such that $(\eta(t_0), \alpha^\vee) \in \mathbb{Z}$ and $(\tau_m(\lambda), \alpha^\vee) < 0$, where m is such that $a_{m-1} < t_0 \leq a_m$.

Denote by Ω^0 the set of L-S paths π such that $\eta(t) = \pi(t)$ for $0 \leq t \leq t_0$, and let Ω be the paths such that either $\pi \in \Omega^0$ or $\eta \geq_t \pi$. Let $\mathcal{V}$ and $\mathcal{V}'$ be the subspaces spanned by the vectors v_π, $\pi \in \Omega$ respectively v_π, $\pi \in \Omega - \Omega^0$. It is easily seen that both subspaces are $U_k^+(\mathfrak{g})$-stable, so $\mathcal{V}^0 := \mathcal{V}/\mathcal{V}'$ is a B-module, with basis the images $\overline{v}_\pi$, $\pi \in \Omega^0$, of the vectors v_π.

We define a twisted action of the operators e_α, f_α on $\Omega^0 \cup \{0\}$. Set $\eta_1 := \eta|_{[0,t_0]}$, i.e., $\eta_1(s) := \eta(st_0)$ for $s \in [0,1]$. A path $\pi \in \Omega^0$ can hence be written as $\pi = \eta_1 * \pi_1$. We define:

$$\tilde{f}_\alpha(\pi) := \eta_1 * (f_\alpha(\pi_1)), \quad \tilde{e}_\alpha(\pi) := \eta_1 * (e_\alpha(\pi_1)).$$

Note that the definition makes sense because $(\eta(t_0), \alpha^\vee) \in \mathbb{Z}$. Further, it is easy to see by [16], Lemma 3.1, that $\tilde{e}_\alpha(\pi), \tilde{f}_\alpha(\pi) \in \Omega^0 \cup \{0\}$. Set

$$\tilde{\ell}(\pi) := (\pi(1), \alpha^\vee) - \min\{(\pi_1(s), \alpha^\vee) | s \in [0,1]\},$$

then the number of nonzero elements in $\dots, \tilde{e}_\alpha^2(\pi), \tilde{e}_\alpha(\pi), \pi, \tilde{f}_\alpha(\pi), \dots$ is $\tilde{\ell}(\pi) + 1$. We call $\tilde{\ell}(\pi)$ the twisted α-string length of π. The same arguments as in the first section apply to see that $\sum_{\pi \in \Omega^0} e^{\pi(1) - \eta(t_0)}$ is the character of an $U_k(\mathfrak{sl}_2(\alpha))$-module.

Lemma 8. *$\mathcal{V}^0$ admits a B-stable filtration such that the unipotent radical of $P(\alpha)$ acts trivially on the subquotients, and the B_α-module structure of the subquotients is induced by an $SL_2(\alpha)$-structure, twisted by the character $\eta(t_0)$.*

Proof. Let $\tilde{R}$ be the ring obtained by adjoining all roots of unity to $\mathbb{Z}$ (section 4). We realize $\mathcal{V}^0_{\tilde{R}}$ as a submodule of $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$, where $\overline{\ell}$ is chosen appropriately as in section 4. For $\tau \in W/W_\lambda$ denote by $N_{\tilde{R}}(\lambda)^0(\tau)$ the $U^+_{q,\tilde{R}}(\mathfrak{g}^t)$-submodule of $N_{\tilde{R}}(\lambda)(\tau)$ spanned by all weight vectors of weight $\neq \tau(\lambda)$. Let M and M' be the $U^+_{\tilde{R}}(\mathfrak{g}^t)$-submodules of $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}}$ spanned by the sum of the following submodules:

$$\begin{aligned} M := & \sum_{j=1}^{m-1} \sum_{i=1}^{\overline{\ell}(a_j - a_{j-1})} N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell} a_{j-1} + i - 1} \otimes N_{\tilde{R}}(\lambda)(\tau_j) \otimes N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1 - a_{j-1}) - i} \\ & + \sum_{i=1}^{\overline{\ell}(t_0 - a_{m-1})} N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell} a_{m-1} + i - 1} \otimes N_{\tilde{R}}(\lambda)(\tau_m) \otimes N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1 - a_{m-1}) - i} \\ M' := & \sum_{j=1}^{m-1} \sum_{i=1}^{\overline{\ell}(a_j - a_{j-1})} N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell} a_{j-1} + i - 1} \otimes N_{\tilde{R}}(\lambda)^0(\tau_j) \otimes N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1 - a_{j-1}) - i} \\ & + \sum_{i=1}^{\overline{\ell}(t_0 - a_{m-1})} N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell} a_{m-1} + i - 1} \otimes N_{\tilde{R}}(\lambda)^0(\tau_m) \otimes N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1 - a_{m-1}) - i} \end{aligned}$$

Note that $\mathcal{V}' \subset M'$ and $\mathcal{V} \subset M$. The quotient M/M' has as basis the images of all tensor products of weight vectors of the form

$$v_{\tau_1}^{\otimes \overline{\ell} a_1} \otimes \dots v_{\tau_m}^{\otimes \overline{\ell}(t_0 - a_{m-1})} \otimes m,$$

where m is an arbitrary weight vector in $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$. Again by looking at the leading terms, one sees that the morphism $\mathcal{V} \to M/M^0$ induces an $U_{q,\tilde{R}}^+(\mathfrak{g})$-equivariant inclusion $\mathcal{V}_{\tilde{R}}^0 \hookrightarrow M/M^0$.

Since $U_{q,\tilde{R}}^+(\mathfrak{g}^t)$ acts trivially on the "first" part of the tensor product of the basis elements of M/M', we can identify M/M' as $U_{q,\tilde{R}}^+(\mathfrak{g}^t)$-module with $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$. The inclusion $\mathcal{V}^0 \hookrightarrow N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$ is hence $U_{q,\tilde{R}}^+(\mathfrak{g})$-equivariant.

Write $\mathcal{V}_{\tilde{R}}^0$ as a direct sum $\mathcal{V}_{\tilde{R}}^{0,1} \oplus \dots \oplus \mathcal{V}_{\tilde{R}}^{0,s}$, where each $\mathcal{V}_{\tilde{R}}^{0,j} = \oplus_{k \in \mathbb{Z}} \mathcal{V}_{\tilde{R}}^0(\mu_j + k\alpha)$ for some weight μ_j. Each of these modules is $U_{\tilde{R}}^+(\mathfrak{sl}_2(\alpha))$-stable, and, by choosing an appropriate numeration of the weights, we may assume that $\oplus_{j \geq i} \mathcal{V}_{\tilde{R}}^{0,j}$ is $U_{\tilde{R}}^+(\mathfrak{g})$-stable for all $i = 1, \dots, s$. Note that the unipotent radical of $P(\alpha)$ operates trivially on $\oplus_{j \geq i} \mathcal{V}_{\tilde{R}}^{0,j} / \oplus_{j \geq i+1} \mathcal{V}_{\tilde{R}}^{0,j}$, and, as $U_{\tilde{R}}^+(\mathfrak{sl}_2(\alpha))$-module, the quotient is isomorphic to $\mathcal{V}_{\tilde{R}}^{0,j}$.

It remains to study the $U_{\tilde{R}}^+(\mathfrak{sl}_2(\alpha))$-module structure of these subquotients. By the isomorphism mentioned above, we may hence assume that $\mathcal{V}_{\tilde{R}}^{0,j} = \mathcal{V}_{\tilde{R}}^0$, and Ω^0 consists only of paths of weight $\mu + k\alpha$, $k \in \mathbb{N}$ (so μ is the "lowest" weight occurring in $\mathcal{V}_{\tilde{R}}^0$).

Note that $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$ is an $U_{q,\tilde{R}}(\mathfrak{g}^t)$-module, so it is in particular a module for $U_{q,\tilde{R}}(\gamma)$ - the quantum group generated by $E_\gamma^{(m)}, F_\gamma^{(m)}$ and the $\left[{K_\gamma;c \atop p}\right]$, where γ is the simple root of $\mathfrak{g}^t$ corresponding to α. So we can consider again $(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$, the sum of weight spaces corresponding to weights that are of the form $\overline{\ell}\mu$, where $\mu \in X^t$ is a weight that is integral for α.

The same arguments as above show that $(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$ has a $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-module structure. Note that the image of $\mathcal{V}_{\tilde{R}}^0$ in $N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)}$ is contained in $(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$, and, looking again at the "leading terms", we see that the image is, as $\tilde{R}$-submodule, a direct summand, with basis given by the $\overline{v}_\pi$. Further, the $U_{\tilde{R}}^+(\mathfrak{sl}_2(\alpha))$-structure on $\mathcal{V}_{\tilde{R}}^0$ is the restriction of the action on $(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$.

So suppose $\pi \in \Omega^0$ is a path such that $\pi(1) = \mu$, the lowest weight occurring in $\mathcal{V}_{\tilde{R}}^0$. Then $\tilde{f}_\alpha \pi = 0$ and $m := \tilde{\ell}(\pi)$, the twisted α-string length, is maximal. Note that $m = -(\mu - \eta(t_0), \alpha^\vee)$.

By $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-representation theory, we know that $X_\alpha^{(m)} \overline{v}_\pi \neq 0$, and, by weight consideration (Ω^0 provides the character of $\mathcal{V}_{\tilde{R}}^0$), $X_\alpha^{(m+1)} \overline{v}_\pi = 0$. By $\mathfrak{sl}_2$-theory, this implies in turn $Y_\alpha \overline{v}_\pi = 0$. It follows that $\overline{v}_\pi, X_\alpha \overline{v}_\pi, \dots, X_\alpha^{(m)} \overline{v}_\pi$ is the basis of an $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-submodule of $\mathcal{V}_{\tilde{R}}^0$, which is a direct summand of $(N_{\mathbb{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$.

Denote by $\mathcal{V}_{\tilde{R}}^{0,m}$ the span of the vectors $X_\alpha^{(j)}\overline{v}_\pi$, $\pi \in \Omega^0$, $\pi(1) = \mu$ and $0 \leq j \leq m$. This is an $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-submodule, and a direct summand of $(N_{\mathbb{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$.

Let Ω^0_{low} be the subset of paths such that $\tilde{f}_\alpha\pi = 0$, and let $\Omega^0_{low}(j)$ be the subset of those of twisted α-string length j $(= m - 2k$ for some $k \in \mathbb{N})$. Suppose we have already defined an $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-submodule $\mathcal{V}_{\tilde{R}}^{0,s+2}$ for the $\Omega^0_{low}(j)$, $j \geq s+2$, which is a direct summand of $(N_{\tilde{R}}(\lambda)^{\otimes \overline{\ell}(1-t_0)})^{\overline{\ell},\alpha}$. Fix a basis B_1 of $\mathcal{V}_{\tilde{R}}^{0,s+2}(\mu+(m-s)\alpha/2)$ and complete it to a basis $B(\mu+(m-s)\alpha/2)$ of $\mathcal{V}_{\tilde{R}}^{0}(\mu+(m-s)\alpha/2)$. By $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$ theory, we know that the vectors $X_\alpha^{(i)}b$, $i = 0, \ldots, s$, $b \in B(\mu + (m-s)\alpha/2)$, are all linearly independent, and, if $b \notin B_1$, then we know by weight considerations that $X_\alpha^{(s+1)}b \in \mathcal{V}_{\tilde{R}}^{0,s+2}$. So after replacing b by a linear combination of b with some element from $\mathcal{V}_{\tilde{R}}^{0,s+2}(\mu+(m-s)\alpha/2)$ if necessary, we may assume that $X_\alpha^{(s+1)}b = 0$. The vectors $b, X_\alpha b, \ldots, X_\alpha^{(s)}b$ provide hence a basis for an $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-submodule, which, by construction, is a direct summand as $\tilde{R}$-module.

This inductive procedure provides the desired $U_{\tilde{R}}(\mathfrak{sl}_2(\alpha))$-module structure, which, by base change, provides the $SL_2(\alpha)$-module structure for any algebraically closed field. $\square$

Remark. The same arguments can be used to prove more generally that for any Schubert variety X, the representation $H^0(G/B, \mathcal{L}_\mu \otimes \widehat{H^0(X, \mathcal{L}_\lambda)})$ admits a good filtration.

References

[1] A. D. Berenstein and A.V. Zelevinsky, *Canonical bases for the quantum group of type A_r and piecewise linear combinatorics*, Duke Math. J. **82** (1996), 473-502.

[2] Gonciulea, N., Lakshmibai, V., *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), 215–248.

[3] A. Joseph, *Quantum Groups and their primitive ideals*, Springer Verlag, New York, 1994..

[4] M. Kashiwara, *Crystalizing the q-analogue of Universal Enveloping algebras*, Commun. Math. Phys. **133** (1990), 249–260.

[5] M. Kashiwara, *Crystal bases of modified quantized enveloping algebras*, Duke Math. J. **73** (1994), 383-414.

[6] M. Kashiwara, *Similarity of crystal bases*, Contemp. Math. **194** (1996), 177–186.

[7] M. Kashiwara and T. Nakashima, *Crystal graphs for the representations of the q-analogue of classical Lie algebras*, J. of Algebra **165** (1994), 295-345.

[8] D. E. Knuth, *Permutations, matrices, and generalized Young tableaux*, Pacific Journal of Math. **34** (1970), 709-727..

[9] V. Lakshmibai, *Tangent spaces to Schubert varieties*, Math. Res. Lett. **2** (1995), 473–477.

[10] V. Lakshmibai, P. Littelmann and P. Magyar, *Path model for Bott-Samelson varieties*, in preparation.

[11] V. Lakshmibai, C.S. Seshadri, *Geometry of G/P V*, J. Algebra **100** (1986), 462–557.

[12] A. Lascoux, B. Leclerc and J.-Y. Thibon, *Crystal graphs and q-analogues of weight multiplicities for the root system A_n*, Lett. Math. Phys. **35** (1995), 359–374.

[13] A. Lascoux and M. P. Schützenberger, *Le monoïde plaxique,*, Quaderni della Ricerca Scientifica del C. N. R., Roma, 1981.

[14] B. Leclerc and J.-Y. Thibon, *The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$*, Electron. J. Combin. **3** (1996).

[15] P. Littelmann, *Contracting modules and Standard Monomial Theory for symmetrizable Kac-Moody algebras*, to appear in *JAMS*.

[16] P. Littelmann, *A Littlewood-Richardson formula for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), 329-346.

[17] P. Littelmann, *Paths and root operators in representation theory*, Ann. Math. **142** (1995), 499-525.

[18] P. Littelmann, *A plactic algebra for semisimple Lie algebras*, Adv. Math **124** (1996), 312–331.

[19] P. Littelmann, *Good filtrations and decomposition rules for representations with standard monomial theory*, J. Reine Angew. Math. **433** (1992), 161–180.

[20] P. Littelmann, *Characters of representations and paths in* $\mathfrak{H}^*$, Representation Theory and Automorphic Forms (T. N. Bailey, A. W. Knapp, eds.), vol. 61, American Mathematical Society, RI., 1997, pp. 29–49.

[21] P. Littelmann, *Standard monomial theory*, Proceedings of the Montreal meeting in 1998.

[22] P. Littelmann, *Crystal graphs and Young tableaux*, Journal of Algebra **175** (1995), 65–87.

[23] G. Lusztig, *Introduction to Quantum groups, Progress in Mathematics*, vol. 110, Birkhäuser Verlag, Boston, 1993.

[24] O. Mathieu, *Filtrations of G–modules*, Ann. Sci. École Norm. Sup. **23** (1990), 625–644.

[25] V. B. Mehta, A. Ramanathan, *Schubert varieties in $G/B \times G/B$*, Comp. Math. **67** (1988), 355–358.

[26] A. Ramanathan, *Equations defining Schubert varieties and Frobenius splitting of diagonals*, J. Publ. Math. Inst. Hautes Etud. Sci. **65** (1987), 61–90.

[27] S. Ramanan, A. Ramanathan, *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), 217–224.

[28] K. N. Raghavan, P. Sankaran, *Monomial bases for representations of classical semisimple Lie algebras*, J. Trans. Groups (to appear).