

THE SHAPE OF THE EMERGING CONDENSATE IN EFFECTIVE MODELS OF CONDENSATION

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Abstract. We consider effective models of condensation where the condensation occurs as time t goes to infinity. We provide natural conditions under which the build-up of the condensate occurs on a spatial scale of $1/t$ and has the universal form of a Gamma density. The exponential parameter of this density is determined only by the equation and the total mass of the condensate, while the power law parameter may in addition depend on the decay properties of the initial condition near the condensation point. We apply our results to some examples, including simple models of Bose-Einstein condensation.

Keywords: Emergence, kinetic equation, quantum particles, Bose-Einstein, House-of-cards model, selection, mutation, singular solution, non-linear partial differential equation, non-equilibrium phenomena.

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1. MOTIVATION AND BACKGROUND

Condensation is an important and interesting phenomenon, which is present in many different physical systems. Loosely speaking, condensation occurs when for a system of many particles, a given relevant quantity has the same value for a macroscopic fraction of those particles. The examples that are relevant to this article are natural selection and mutation, where the particles are individuals, the relevant quantity is the fitness, and the condensation occurs at the maximal fitness; and Bose-Einstein condensation, where the particles are Bosons, the relevant quantity is the energy, and the condensation occurs at the minimal quantum energy level. We are interested in what a condensing system looks like when it is dynamically close to condensation.

We will not deal with the difficult issue of condensation in many-particle systems itself, but will instead investigate effective models and manifestations of condensation in a scalar variable. These models are then given in the form of non-linear measure-valued equations in $1+1$ dimensions, the first of the variables being time, and the other the scalar quantity mentioned above. More precisely, for $t \in \mathbb{R}_0^+$, the finite measure p_t describes the concentration of particles at relevant quantity x , and $(p_t)_{t \geq 0}$ solves the equation

$$\partial_t p_t(dx) = F(x, p_t). \tag{1.1}$$

with some functional F . We will give three concrete examples of such systems below: Kingman's model of selection and mutation [11], an approximate model for Bose-Einstein condensation due to Buffet, de Schmedt and Pulé [1] (henceforth called the BSP-model), and a model for bosons in a heat bath investigated by Escobedo, Mischler and Velazquez [7, 8, 9] (referred to als EMV-model below). Another effective model of condensation is the Boltzmann-Nordheim equation [12, 13, 15, 10], but as we will see below, it is too singular for our theory to apply.

In the effective models, condensation is characterized by the behavior of p_t as $t \rightarrow t^*$, where $t^* \leq \infty$ is the time at which condensation occurs. Typically, one assumes that the initial condition p_0 is absolutely continuous with respect to the Lebesgue measure, and t^* is the infimum over all times where $p_t(dx)$ is not absolutely continuous with respect to the Lebesgue measure. If one finds that there exist $\rho > 0$ and $q \in L^1$ so that for suitable test functions ϕ ,

$$\lim_{t \uparrow t^*} \int p_t(dx) \phi(x) = \rho \phi(x^*) + \int q(x) \phi(x) dx, \quad (1.2)$$

this is paraphrased by saying that condensation occurs at $x = x^*$, at time $t = t^*$. We then refer to ρ as the *mass of the condensate*, while q is the *bulk density*. The approach to the limit in Equation (1.2) can be interpreted as the formation of an approximate Dirac distribution around x^* when t is near t^* . The topic of our paper is to investigate in the case $t^* = \infty$ the asymptotic shape of this approximate Dirac distribution in the correct scale.

In cases where $t^* < \infty$, much less is known. The best results that we are aware of are those by Escobedo and Velazquez for the Boltzmann-Nordheim equation [10]. They show that under suitable assumptions on the mass of the initial condition, the solution $p_t(dx)$ to (1.1) has a Lebesgue density \tilde{p}_t that explodes in finite time, i.e. there exists a $t_*^- < \infty$ so that

$$\liminf_{t \uparrow t_*^-} \|\tilde{p}_t\|_\infty = \infty.$$

They also show that there exists a $t_*^+ \geq t_*^-$ so that the weak solution of (1.1) contains a Dirac mass at the condensation point $x^* = 0$ for some $t \in [t_*^-, t_*^+]$. What they cannot show is that the infimum over all possible t_*^+ equals t_*^- , and so an equation like (1.2) is currently not known. We are not aware of any natural examples where (1.2) has been proved for some $t^* < \infty$, and therefore stick to the case $t^* = \infty$.

To our knowledge, the following rigorous results about the asymptotic shape of such emerging condensates exist: In [5], it is shown that for Kingman's model of selection and mutation, the shape is the one of a Gamma-distribution. The same result is found in [9] for a special case of the EMV-model. They also find the influence of the initial condition, depending on its behavior near the condensation point, that will appear in our results below. The results rely on explicit solution formulas for all times, although in [9] a formal asymptotic expansion is used in order to cover also cases where such a solution formula is missing.

The contribution of our paper is to provide conditions which are easy to check, do not require the knowledge of a solution formula, and are sufficient to conclude both the Gamma shape of the near condensate and the possible dependence of that shape on the initial condition. Our conditions are natural in that they only require knowledge about the right hand side of (1.2) when p_t is very close to the formal stationary solution $\rho \delta_{x^*} + q$. Precise statements follow in the next section.

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2. ASSUMPTIONS AND MAIN RESULT

We treat models where condensation occurs at the boundary of the set of possible values of the scalar quantity x , and we normalize these models so that this value is $x = 0$. Further we restrict attention to the case where $t^* = \infty$. Let $\mathcal{M}_0 := \{\rho \delta_0 + p dx : \rho \geq 0, p \in L^1(\mathbb{R}_0^+)\}$

be the subspace of the space of finite measures on \mathbb{R}_0^+ that have a density with respect to Lebesgue measure except possibly at the origin where we allow for a Dirac measure δ_0 with mass ρ . Fix $\alpha > 0$, and let $\mathbf{B} : \mathcal{M}_0 \rightarrow C(\mathbb{R}_0^+)$ and $\mathbf{C} : \mathcal{M}_0 \rightarrow C(\mathbb{R}_0^+)$ be (not necessarily linear) operators. Consider the equation

$$\partial_t p_t(dx) = \mathbf{B}[p_t]p_t(dx) + x^\alpha \mathbf{C}[p_t] dx. \quad (2.1)$$

An element $p \in \mathcal{M}_0$ is called *stationary*, if

$$\mathbf{B}[p]p(dx) + x^\alpha \mathbf{C}[p] dx = 0.$$

In most cases we consider evolutions where p_t does not have an atom in 0. In this case we refer to the Lebesgue density of p_t by the same symbol p_t . With this convention the equation reads

$$\partial_t p_t(x) = \mathbf{B}[p_t]p_t(x) + x^\alpha \mathbf{C}[p_t].$$

When comparing (2.1) to the most general equation (1.2), we see that we demand a decomposition of the right hand side into a homogenous part and a remainder. While such a decomposition can always be achieved (e.g. by setting $\mathbf{B} = 0$), the restriction lies in the assumed regularity of the images under \mathbf{B} and \mathbf{C} . In particular, we assume that when dividing the inhomogenous part by a factor of x^α , we still retain a function that is bounded at $x = 0$. In all the concrete and relevant examples that we are aware of, the decomposition is unique and easy to find. In the Boltzmann-Nordheim model, however, measures with a Dirac mass at the origin are too singular for the equation to make sense in a classical way [10], and so a decomposition like (2.1) with the corresponding regularity assumptions fails.

Definition: We say that a solution $(p_t)_{t \geq 0}$ to (2.1) *converges regularly* to an element $p_\infty \in \mathcal{M}_0$ if

- (i): $p_t \rightarrow p_\infty$ weakly as $t \rightarrow \infty$ as measures;
- (ii): the following two equations hold:

$$\lim_{t \rightarrow \infty} \|\mathbf{B}[p_t] - \mathbf{B}[p_\infty]\|_{C^1([0, \delta])} = 0, \quad (2.2)$$

$$\lim_{t \rightarrow \infty} \|\mathbf{C}[p_t] - \mathbf{C}[p_\infty]\|_{C([0, \delta])} = 0. \quad (2.3)$$

Here, $\|f\|_{C^1([0, \delta])} = \sup\{|f(x)| + |f'(x)| : 0 \leq x \leq \delta\}$, and $\|f\|_{C([0, \delta])} = \sup\{|f(x)| : 0 \leq x \leq \delta\}$.

When p_∞ is a measure with positive condensate mass, (i) above is what is usually proved when condensation is shown, see e.g. [8, 1]. (ii) is more particular to our needs. Note that (2.2) demands that the difference of $\mathbf{B}[p_t]$ and $\mathbf{B}[p_\infty]$ is of higher regularity than each individual term needs to be.

In two of the examples that we will give, the operators \mathbf{B} and \mathbf{C} are affine integral operators. In both of those examples, it has been shown that the convergence of p_t to p_∞ is in L^1 away from $x = 0$. The next proposition states that in such cases, (2.2) and (2.3) already follow from natural regularity assumptions on the integral kernels of \mathbf{B} and \mathbf{C} .

Proposition 2.1. *Assume that \mathbf{B} and \mathbf{C} are affine integral operators, i.e.*

$$\mathbf{B}[p](x) = \int_0^\infty K_b(x, y)p(dy) + R_b(x), \quad \mathbf{C}[p](x) = \int_0^\infty K_c(x, y)p(dy) + R_c(x),$$

where K_b , K_c , and $\partial_x K_b$ are elements of $C_b([0, \delta] \times \mathbb{R}_0^+)$ for some $\delta > 0$, while R_b and R_c are arbitrary functions.

Let (p_t) be a solution to (2.1) with initial condition $p_0(x)dx$, i.e. without an atom at zero. Assume that p_t converges weakly to $p_\infty = \rho\delta_0 + q(x)dx$ with $\rho \geq 0$ and $q \in L^1$, and in addition assume that

$$\lim_{t \rightarrow \infty} \int_\delta^\infty |p_t(x) - q(x)| dx = 0 \quad (2.4)$$

for all $\delta > 0$. Finally, assume that there exists $\tilde{\delta} > 0$ such that

$$\limsup_{t \rightarrow \infty} \int_0^{\tilde{\delta}} |p_t(x)| dx < \infty. \quad (2.5)$$

Then p_t converges regularly to p_∞ .

The proof consists of standard applications of integral convergence theorems. We give it in the appendix for the convenience of the reader. Note that since R_b and R_c drop out when considering differences like $\mathbf{B}[p_t](x) - \mathbf{B}[p_\infty](x)$, they are indeed arbitrary, although the regularity requirements preceding (2.1) will usually mean that they need to be continuous and bounded.

Now we state our main result.

Theorem 2.2. *Assume that (p_t) solves equation (2.1) with initial condition $p_0(x)dx$, i.e. without atom at zero. For the density p_0 , assume that there exists $\alpha_0 > 0$ and a function $\eta: \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$, which is continuous and positive at zero, so that*

$$p_0(x) := x^{\alpha_0} \eta(x).$$

Assume further that (p_t) converges regularly to a stationary limit $p_\infty = \rho\delta_0 + q(x)dx$ with $\rho > 0$ and $q \in L^1$. Finally, assume that for this limit,

$$c_1 := \mathbf{C}[p_\infty](0) > 0, \quad (2.6)$$

and that, for some $\alpha > 0$,

$$c_2 := \lim_{x \rightarrow 0} \frac{x^{\alpha-1}}{q(x)} > 0 \text{ exists.} \quad (2.7)$$

Then with $\gamma := c_1 c_2$ and $\beta := \min\{\alpha, \alpha_0\}$ we have, uniformly on compact intervals of \mathbb{R}_0^+ , that

$$\lim_{t \rightarrow \infty} \frac{1}{t} p_t\left(\frac{x}{t}\right) = C e^{-\gamma x} x^\beta. \quad (2.8)$$

Above, $C = \rho\gamma^\beta / \Gamma(\beta)$, i.e. C is such that the right hand side of (2.8) integrates to ρ .

The main feature of this result is the universal nature of the gamma shape of the emerging condensate. In all examples that we are aware of, the stationary limit p_∞ in Theorem 2.2 only depends on the mass of the initial condition, but not on its shape. In these cases, the statement can also be read as the following dichotomy: either $\alpha \leq \alpha_0$, in which case the initial condition is irrelevant for the shape of the emerging condensate; or, $\alpha_0 < \alpha$, in which case the exponential decay of the Gamma distribution is still governed by the constant from (2.7), but the power law near $x = 0$ is the same as in the initial condition. From the calculations in our proofs, it is apparent that the second case could be strengthened in the following way: if the initial condition dominates the inhomogeneity near $x = 0$, the emerging condensate will look like the initial condition at an appropriate scale. In particular, one might think of initial conditions that switch between different power laws α_1 and α_2 infinitely often in the approach to zero. In that case, we would have no convergence in (2.8), but rather an oscillating behavior. Since this case does not seem very relevant and would need rather careful statements and investigations, we do not pursue it any further.

Note that condition (2.7) can alternatively be read as a condition on $\mathbf{B}[p_\infty](x)$ near the point $x = 0$. Namely, by stationarity of p_∞ we have that

$$\mathbf{B}[p_\infty](x) = -\frac{x^\alpha}{q(x)}\mathbf{C}[p_\infty](x)$$

for Lebesgue almost all $x \geq 0$ and by continuity this holds for *all* $x \geq 0$. For $x = 0$ the right hand side equals zero so that $\mathbf{B}[p_\infty](0) = 0$. Thus, condition (2.7) means that $x \mapsto \mathbf{B}[p_\infty](x)$ is differentiable at $x = 0$, and that its derivative equals $-\gamma$.

3. EXAMPLES

3.1. Selection mutation equations. A natural set of examples for our theory arise from equations describing the fitness distribution of a population evolving by selection and mutation. We focus on Kingman's model [11] and briefly mention some generalizations and variants at the end of this section.

Kingman's model of selection and mutation is originally framed in discrete time, see [11]. We start with an initial fitness distribution $p_0(dx)$ of a diffuse population, which is a probability measure on $(0, 1)$. By $p_n(dx)$ we denote the fitness distribution in the n -th generation. It satisfies the recursion

$$p_{n+1}(dx) = (1 - \beta)\frac{x}{w_n}p_n(dx) + \beta r(dx),$$

where $w_n = \int_0^1 xp_n(dx)$ is the mean fitness at generation n , r is the fitness distribution for spontaneous mutations, and $\beta \in (0, 1)$ is the frequency of mutation. We assume that r is a probability measure on $(0, 1)$ with essential supremum at $x = 1$. Kingman's idea is that a proportion $1 - \beta$ of the population is selected from the previous generation with a selective advantage proportional to their fitness, and a proportion β of the population experiences mutation, which destroys the individuals' biochemical 'house of cards' so that the mutant fitness distribution r does not depend on their previous fitness. Kingman showed that condensation at the maximal fitness $x = 1$ occurs if

$$\beta \int_0^1 \frac{r(dx)}{1-x} < 1. \quad (3.1)$$

To adapt the model to our framework, we switch to a continuous time model and change variables so that condensation occurs at $x = 0$. The result is the equation

$$\partial_t p_t(dx) = \left((1 - \beta)\frac{1-x}{w[p_t]} - 1 \right) p_t(dx) + \beta u(dx), \quad (3.2)$$

where $w[p_t] = \int_0^1 (1-x)p_t(dx)$ and $u(dx) = r(d(1-x))$. Kingman's arguments show that if

$$\beta \int_0^1 \frac{u(dx)}{x} < 1$$

we have $p_t \rightarrow p_\infty$ weakly for the stationary solution p_t given by

$$p_\infty(dx) = \left(1 - \beta \int_0^1 \frac{u(dx)}{x} \right) \delta_0 + \beta \frac{u(dx)}{x}.$$

We assume that $u(x) = x^\alpha u_0(x)$, for some $\alpha > 0$, and $u_0(x)$ continuous and strictly positive at $x = 0$. Then,

$$\mathbf{B}[p] = (1 - \beta)\frac{1-x}{w[p]} - 1,$$

and $\mathbf{C}[p] = \beta u_0$. In particular (2.2) and (2.3) are trivially fulfilled. Moreover,

$$w[p_\infty] = \left(1 - \beta \int_0^1 \frac{u(dx)}{x}\right) + \beta \int \frac{1-x}{x} u(dx) = 1 - \beta,$$

as $u(dx)$ is a probability measure. Then, $\mathbf{B}[p_\infty](x) = -x$, and $\mathbf{C}[p_\infty](x) = \beta u_0(x)$, confirming that p_∞ is stationary. Moreover, the bulk density $q(x)$ is given by $\beta u(x)/x = \beta x^{\alpha-1} u_0(x)$. Thus, c_2 from (2.7) is given by $1/\beta u_0(0)$, and Theorem 2.2 holds with $\gamma = 1$. We thus obtain the gamma-shape of the emerging condensate under less restrictive conditions than Dereich and Mörters [5].

Kingman's model was generalized by Yuan [16] to model the long-term evolution of *Escherichia.coli* in the Lenski experiment. This model can still be fitted to our framework. Park and Krug [14] generalize Kingman's model to unbounded fitness distributions, which leads to a qualitatively different emergence of a condensate (at infinity). They also investigate the relation of Kingman's model to a stochastic finite population model. Dereich [4] identifies the shape of the emergent condensate in a random network model with fitness.

We now discuss in more detail a stochastic particle model based on a branching process closely related to Kingman's original model, which is investigated in [6]. In this model immortal particles produce offspring with a rate given by their fitness. Independently, each offspring particle is a mutant with probability β and otherwise inherits the parent's fitness. Mutants receive their fitness by sampling from the distribution r . The expected fitness distribution q_t of particles alive at time t therefore satisfies

$$\dot{q}_t(dx) = (1 - \beta)xq_t(dx) + \beta \int y q_t(dy)r(dx).$$

Taking $a_t = \int_0^t \int y q_s(dy) ds$ the normalized and time-changed quantity

$$\tilde{p}_t = \frac{q_{a_t^{-1}}}{\int q_{a_t^{-1}}(dy)}$$

therefore satisfies

$$\dot{\tilde{p}}_t(dx) = \left((1 - \beta) \frac{x}{\int y \tilde{p}_t(dy)} - 1 \right) \tilde{p}_t(dx) + \beta r(dx).$$

Moving now the condensation point to the origin we are back in Kingman's model and Theorem 2.1 can be applied. If (3.1) holds we have $a_t \sim (1 - \beta)t$ and we observe that also in the stochastic model the condensate is forming on the scale $1/t$ in expectation. The behaviour of this stochastic model in probability is more difficult to investigate and largely open.

3.2. A model of Bosons in contact with a bath of Fermions. In a series of papers [7, 8, 9], Escobedo, Mischler and Velazquez study an effective model for Bosons in contact with a bath of Fermions in thermal equilibrium. Given a function $b : \mathbb{R}_+^0 \times \mathbb{R}_+^0 \rightarrow \mathbb{R}_+$ of the form

$$b(x, y) = e^{\eta x} e^{\eta y} \sigma(x, y)$$

with $\eta \in [0, 1)$ and $0 \leq \sigma \in L^\infty(\mathbb{R}_+^0 \times \mathbb{R}_+^0)$ symmetric, one asks for solutions $(F_t)_{t \geq 0}$ of the equation

$$\partial_t F_t(x) = \int_0^\infty b(x, y) \left(F_t(y)(x^2 + F_t(x))e^{-x} - F_t(x)(y^2 + F_t(y))e^{-y} \right) dy.$$

Note that the right hand side is well defined as long as F_t is in the space $L^1(\mathbb{R}_0^+, e^{\eta x} dx)$. To apply our results we consider the transformed solution $(p_t)_{t \geq 0}$ given by $p_t(x) = e^{\eta x} F_t(x)$. It satisfies as $L^1(\mathbb{R}_+^0)$ -valued system the equation

$$\partial_t p_t(x) = e^{\eta x} \int_0^\infty \sigma(x, y) \left(p_t(y)(x^2 e^{\eta x} + p_t(x)) e^{-x} - p_t(x)(y^2 e^{\eta y} + p_t(y)) e^{-y} \right) dy. \quad (3.3)$$

The extension of this equation to elements of \mathcal{M}_0 is straightforward, and it fits into the framework of (2.1) with $\alpha = 2$,

$$\mathbf{B}[p](x) = e^{\eta x} \int_0^\infty p(dy) \sigma(x, y) (e^{-x} - e^{-y}) - e^{\eta x} \int_0^\infty \sigma(x, y) y^2 e^{(\eta-1)y} dy, \quad (3.4)$$

and

$$\mathbf{C}[p](x) = e^{(2\eta-1)x} \int_0^\infty \sigma(x, y) p(dy). \quad (3.5)$$

Under suitable assumptions on σ and the initial condition, Escobedo and Mischler [7, 8] show existence of solutions for all times, and convergence of the solution as $t \rightarrow \infty$ towards the stationary solution. In the case where $\int_0^\infty F_0(x) dx > \int_0^\infty \frac{x^2}{e^x - 1} dx = 2.40411\dots$ condensation occurs and the limit is of the form

$$F_\infty(dx) = \rho \delta_0 + Q(x) dx \quad \text{with } Q(x) = \frac{x^2}{e^x - 1}$$

and $\rho = \int_0^\infty F_0(x) dx - \int \frac{x^2}{e^x - 1} dx$. In that case (p_t) has as limit

$$p_\infty(dx) = \rho \delta_0 + q(x) dx \quad \text{with } q(x) = \frac{x^2 e^{\eta x}}{e^x - 1}. \quad (3.6)$$

In [9], the authors investigate the shape of the emerging condensate; they show rigorously (by means of an explicit solution formula) that the emerging condensate is Gamma-shaped in the special case that $b(x, y) \equiv 1$. Moreover, they find that the power law parameter of the Gamma distribution depends on how the initial condition vanishes at $x = 0$; this corresponds to Theorem 2.2 of the present paper. Very interestingly, they also obtain asymptotics for the case where the initial condition already has a Dirac mass at $x = 0$. In that case, there is no emerging condensate as $t \rightarrow \infty$, instead the Dirac mass itself grows to its final value (determined by the total mass of the initial condition) as $t \rightarrow \infty$; see Theorem 1, part (ii) of [9]. This result shows that we cannot drop the assumption on absolute continuity of the initial condition in Theorem 2.2.

We will now use our general theory in order to find more general conditions on b so that the Gamma shape of the emerging condensate still holds. We first assume that $(x, y) \mapsto \sigma(x, y)$ is continuous, continuously differentiable with respect to x for all y , and that, for some $\delta > 0$,

$$\sup\{|\partial_x \sigma(x, y)| : 0 \leq x \leq \delta, y \in \mathbb{R}^+\} < \infty. \quad (3.7)$$

This condition is tailor-made for the assumptions on \mathbf{B} and \mathbf{C} of Proposition 2.1. In fact it is slightly more than we really need, since we have

$$K_b(x, y) = \sigma(x, y)(e^{-x} - e^{-y}),$$

and thus at $x = y$ differentiability of σ is not needed due to the presence of the factor $e^{-x} - e^{-y}$. We further assume that

$$\int_0^\infty \sigma(0, y) p_\infty(dy) > 0. \quad (3.8)$$

This assumption ensures the validity of conditions (2.6) and (2.7). Indeed, we compute

$$\mathcal{C}[p_\infty](x) = e^{(2\eta-1)x} \left(\rho\sigma(x,0) + \int_0^\infty \sigma(x,y) \frac{y^2 e^{\eta y}}{e^y - 1} dy \right),$$

and thus $\mathcal{C}[p_\infty](0) > 0$, so (2.6) holds. Since

$$\lim_{x \rightarrow 0} \frac{x^{\alpha-1}}{q(x)} = \lim_{x \rightarrow 0} e^{\eta x} \frac{e^x - 1}{x} = 1,$$

we also have (2.7), and find that $\gamma = \int \sigma(0,y)p_\infty(dy)$. Note that (3.8) is weaker than the assumption $\sigma(0,0) > 0$ that was made in the appendix of [9], where non-constant b are treated non-rigorously, using matched asymptotic expansions.

Finally we need to verify (2.4). This is proved by Escobedo and Mischler in Theorem 6 of [8] for the case $\eta = 0$ and we believe that the arguments given there allow to prove (2.4) also for arbitrary values of $\eta > 0$. In conclusion, if conditions (3.7) and (3.8) hold and $\eta = 0$, then Theorem 2.2 holds with $\alpha = 2$ and $\gamma = \int \sigma(0,y)p_\infty(dy)$.

3.3. Kinetics of Bose-Einstein condensation. Our final example is a simple model for the emergence of a condensate in a Bose gas in contact with a heat bath, which was developed by Buffet, de Smedt and Pulé in [1, 2].

Let $\hat{C} : \mathbb{R} \rightarrow \mathbb{R}_0^+$ be a strictly positive, bounded function satisfying the *KMS relation*

$$\hat{C}(-z) = \hat{C}(z) e^{\beta z} \tag{3.9}$$

for some $\beta > 0$. Physically, \hat{C} is the Fourier transform of the heat bath correlation function, and $\beta > 0$ is the inverse temperature of the heat bath. Note also that the assumption that \hat{C} is bounded and (3.9) imply exponential decay of $\hat{C}(z)$ as $z \rightarrow \infty$.

We also define

$$A(z) = \hat{C}(-z) - \hat{C}(z) = \hat{C}(z)(e^{\beta z} - 1),$$

$$F(x) = \frac{x^{1/2}}{\sqrt{2}\pi^2}.$$

F is the density of states in the case of a Bose gas in a 3-dimensional box. In order to model the Bose gas in other environments we would need to modify F ; for example, for the Bose gas in a 3-dimensional harmonic trap, F would decay like $x^{3/2}$ near $x = 0$; see also the discussion in [3]. In what follows we will consider the more general form

$$F(x) = f(x)x^\alpha, \quad \alpha > 0, \quad f(0) > 0. \tag{3.10}$$

The energy distribution p_t of the Bose gas at time $t \geq 0$ then satisfies the equation

$$\begin{aligned} \partial_t p_t(x) = & \int_0^\infty A(y-x)p_t(x)p_t(y) dy \\ & - \int_0^\infty \hat{C}(y-x)F(y)p_t(x) dy + \int_0^\infty \hat{C}(x-y)p_t(y)F(x) dy. \end{aligned} \tag{3.11}$$

It is easy to check that (3.11) preserves the total mass $\int_0^\infty p_t(x) dx$ for all times $t \geq 0$. It has been shown in [1] that there is a global solution to (3.11) and moreover, there exists $\rho_c > 0$ such that, for all initial energy densities satisfying

$$\int_0^\infty p_0(x) dx > \rho_c + \rho$$

for some $\rho > 0$, (p_t) converges weakly to the stationary solution

$$p_\infty(dx) = \rho\delta_0 + q(x) dx$$

with bulk density

$$q(x) = \frac{F(x)}{e^{\beta x} - 1}.$$

Moreover, Theorem 2 of [1] states that

$$\lim_{t \rightarrow \infty} \int_c^\infty |p_t(y) - q(y)| dy = 0$$

for all $c > 0$. Thus, (2.4) and (2.5) hold.

The decomposition of equation (3.11) according to (2.1) is given by

$$\begin{aligned} \mathbf{B}[p](x) &= \int_0^\infty A(y-x)p(y) dy - \int_0^\infty \hat{C}(y-x)F(y) dy, \\ \mathbf{C}[p](x) &= f(x) \int_0^\infty \hat{C}(x-y)p(y) dy. \end{aligned}$$

So in the context of Proposition 2.1, we have $K_b(x, y) = A(y-x)$, $K_c(x, y) = \hat{C}(x-y)$, $R_1(x) = \int_0^\infty \hat{C}(y-x)F(y) dy$ and $R_2(x) = 0$.

We now assume that the restriction of \hat{C} to the interval $(-\infty, 0)$ is continuously differentiable with a bounded derivative, and that f is continuous. Then the integral kernels $K_b(x, y) = A(y-x)$ and $K_c(x, y) = \hat{C}(x-y)$ are bounded and continuous. In particular, \mathbf{B} and \mathbf{C} map \mathcal{M}_0 to $C(\mathbb{R}_0^+)$. Moreover, $x \mapsto A(y-x) = \hat{C}(x-y) - \hat{C}(y-x)$ is continuously differentiable with bounded derivative whenever $x \neq y$. For the case $x = y$, note that $A(z) = \hat{C}(z)(e^{\beta z} - 1)$, and thus $x \mapsto A(x-y)$ is differentiable also at $x = y$ with derivative $\beta\hat{C}(0)$. Altogether, we get that $\partial_x K_b(x, y)$ is in $C_b([0, \delta] \times \mathbb{R}_0^+)$, and Proposition 2.1 implies that p_t converges regularly to p_∞ .

It remains to check assumptions (2.6) and (2.7). Clearly,

$$\mathbf{C}[p_\infty](0) = f(0) \int \hat{C}(y)p_\infty(y) dy > 0,$$

which shows that α is determined by the choice in (3.10). Furthermore,

$$\lim_{x \rightarrow 0} \frac{x^{\alpha-1}}{q(x)} = \lim_{x \rightarrow 0} \frac{e^{\beta x} - 1}{x f(x)} = \frac{\beta}{f(0)},$$

which shows that

$$\gamma = \beta \int_0^\infty \hat{C}(y)p_\infty(y) dy.$$

By Theorem 2.2, we conclude again that the emerging condensate is described by a Gamma distribution on the relevant scale.

4. PROOF OF THE THEOREM

Let (p_t) be a solution to (2.1). We define, for $x \geq 0$,

$$b_t(x) := \mathbf{B}[p_t](x),$$

and, for $x > 0$,

$$c_t(x) = \mathbf{C}[p_t](x).$$

Then p_t solves the time-inhomogenous linear equation

$$\partial_t p_t(x) = b_t(x)p_t(x) + x^\alpha c_t(x)$$

with initial condition p_0 , and thus has the representation

$$p_t(x) = \int_0^t e^{\int_s^t b_u(x) du} x^\alpha c_s(x) ds + e^{\int_0^t b_s(x) ds} p_0(x). \quad (4.1)$$

We will re-write this representation in a form that will be convenient later on, using the following definitions:

$$\begin{aligned} W_t &:= e^{\int_0^t b_s(0) ds}, \\ \gamma_t(x) &:= -\frac{1}{x}(b_t(x) - b_t(0)) \quad \text{for } x > 0, \\ \bar{\gamma}_{s,t}(x) &:= \begin{cases} \frac{1}{t-s} \int_s^t \gamma_r(x) dr & \text{if } t > s \\ \gamma_t(x) & \text{if } t = s. \end{cases} \end{aligned}$$

Then we have

$$p_t(x) = \int_0^t \frac{W_t}{W_s} x^\alpha c_s(x) e^{-(t-s)x\bar{\gamma}_{s,t}(x)} ds + W_t e^{-tx\bar{\gamma}_{0,t}(x)} p_0(x), \quad (4.2)$$

Next, we use our assumptions in order to prove some properties of the quantities c_t and γ_t .

Proposition 4.1. *Let the assumptions of Theorem 2.2 be fulfilled. Then*

- (1) $\lim_{t \rightarrow \infty} b_t(0) = 0$.
- (2) *For sufficiently large t , $x \mapsto \gamma_t(x)$ can be continuously extended to all $x \geq 0$. Further, there exists $\delta > 0$ and a strictly positive function $\gamma_\infty : [0, \delta] \rightarrow \mathbb{R}^+$ with the property that*

$$\lim_{t \rightarrow \infty} \sup_{x \in [0, \delta]} |\gamma_t(x) - \gamma_\infty(x)| = 0.$$

Moreover, we have that $\gamma_\infty(0) = \gamma$, the latter being defined in (2.7).

- (3) *There exists $\delta > 0$ and a continuous function $c_\infty : [0, \delta] \rightarrow \mathbb{R}$ such that $c_\infty(0) > 0$, and*

$$\lim_{t \rightarrow \infty} \sup_{x \in [0, \delta]} |c_t(x) - c_\infty(x)| = 0.$$

Proof. By the comments at the end of Section 2 we have $\mathbf{B}[p_\infty](0) = 0$ so that by (2.2)

$$\lim_{t \rightarrow \infty} b_t(0) = \lim_{t \rightarrow \infty} \mathbf{B}[p_t](0) = \mathbf{B}[p_\infty](0) = 0,$$

and (1) is shown. For (2), we define

$$\gamma_\infty(x) := -\frac{1}{x} \mathbf{B}[p_\infty](x) \quad \text{for } x > 0.$$

Since $\mathbf{B}[p_\infty](x)q(x) = -x^\alpha \mathbf{C}[p_\infty](x)$ for all $x > 0$, we find that

$$\lim_{x \rightarrow 0} \gamma_\infty(x) = \lim_{x \rightarrow 0} \frac{x^\alpha}{xq(x)} \mathbf{C}[p_\infty](x) = \gamma$$

by Assumption (2.7) and the definition of γ . In particular, γ_∞ can be continuously extended to the whole half line. We write b_∞ for $\mathbf{B}[p_\infty]$, and put

$$s_t(x) := b_\infty(x) - b_t(x).$$

Then by (2.2), $x \mapsto s_t(x)$ is an element of $C^1[0, \delta]$ for some $\delta > 0$ and all sufficiently large t , and $\lim_{t \rightarrow \infty} \|s_t\|_{C^1([0, \delta])} = 0$. Furthermore,

$$\gamma_t(x) - \gamma_\infty(x) = \frac{1}{x}(s_t(x) - s_t(0)) = \int_0^1 s'_t(rx) dr,$$

and thus (2) follows. (3) follows directly from Assumption (2.3). \square

Recall our assumption on the initial condition p_0 : there exists $\alpha_0 > 0$ and $\eta \in L^1$ so that η is continuous at $x = 0$, $\eta(0) > 0$, and

$$p_0(x) = x^{\alpha_0} \eta(x) \quad (4.3)$$

for all $x \in [0, \delta]$. Let us also define, for $\beta > 0$,

$$Q_t(\beta) := W_t^{-1}(t+1)^{1+\beta}. \quad (4.4)$$

A direct calculation then gives an expression for p_t/t at arguments of the order $1/t$, namely

$$\frac{Q_t(\beta)}{t} p_t\left(\frac{x}{t}\right) = \left(\frac{t+1}{t}\right)^{1+\beta} e^{-x\gamma_\infty(0)} \left(t^{\beta-\alpha} x^\alpha J(x, t) + t^{\beta-\alpha_0} x^{\alpha_0} e^{x(\gamma_\infty(0) - \bar{\gamma}_{0,t}(x/t))} \eta\left(\frac{x}{t}\right) \right), \quad (4.5)$$

with

$$J(x, t) = \int_0^t \frac{Q_s(\beta)}{(s+1)^{\beta+1}} c_s(x/t) e^{x(\gamma_\infty(0) - \frac{t-s}{t} \bar{\gamma}_{s,t}(x/t))} ds.$$

Now it is easy to prove the following

Proposition 4.2. *Assume that the quantities (b_t) and (c_t) have the properties (1) - (3) from Proposition 4.1, and assume in addition that for $\beta := \min\{\alpha, \alpha_0\}$, the limit $Q_\infty = \lim_{t \rightarrow \infty} Q_t(\beta)$ exists in $(0, \infty)$. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} p_t\left(\frac{x}{t}\right) = \frac{1}{Q_\infty} e^{-x\gamma_\infty(0)} \left(1_{\{\beta=\alpha\}} x^\alpha \int_0^\infty \frac{Q_s(\beta)}{(s+1)^\beta} c_s(0) ds + 1_{\{\beta=\alpha_0\}} x^{\alpha_0} \eta(0) \right), \quad (4.6)$$

and the limit is uniform in x on compact intervals in $(0, \infty)$.

Proof. First we analyse the first summand on the right hand side of (4.5). We show that $J(x, t)$ converges uniformly to

$$\int_0^\infty \frac{Q_s(\beta)}{(s+1)^\beta} c_s(0) ds.$$

To see this, note that by Proposition 4.1, $\bar{\gamma}_{0,t}$ converges uniformly on $[0, \delta]$ to γ_∞ and by continuity of γ_∞ in zero we get local uniform convergence of $\bar{\gamma}_{0,t}(x/t)$ to $\gamma_\infty(0)$. Hence for each fixed s , the quantity

$$D(s, t) := \sup_{0 \leq x \leq K} \left| c_s(x/t) e^{x(\gamma_\infty(0) - \frac{t-s}{t} \bar{\gamma}_{s,t}(x/t))} - c_\infty(0) \right|$$

converges to zero as $t \rightarrow \infty$. The integrand in $J(x, t)$ and $\frac{Q_s(\beta)}{(s+1)^\beta} c_s(0)$ are both bounded by a constant multiple of the integrable function $(s+1)^{-1-\beta}$, uniformly in x , on compact intervals. Thus for $M > 0$,

$$\sup_{0 \leq x \leq K} \left| J(x, t) - \int_0^\infty \frac{Q_s(\beta)}{(s+1)^\beta} c_s(0) ds \right| \leq \int_0^M \frac{Q_s(\beta)}{(s+1)^\beta} D(s, t) ds + C \int_M^\infty (s+1)^{-1-\beta} ds.$$

The first term above converges to zero by dominated convergence, as $t \rightarrow \infty$. By taking $M \rightarrow \infty$ afterwards, the asymptotic of the first term follows. Using that, uniformly on compact intervals, $e^{x(\gamma_\infty(0) - \bar{\gamma}_{0,t}(x/t))} \eta\left(\frac{x}{t}\right) \rightarrow \eta(0)$ we get convergence of the second term.

□

We have just proved Theorem 2.2 under the additional assumption that $\lim_{t \rightarrow \infty} Q_t(\beta)$ exists and is strictly positive. Since the definition of $Q_t(\beta)$ involves the function $b_t(0) = \mathbf{B}[p_t](0)$ and thus the solution itself, such a condition is not desirable. The main step of our proof consists in showing that existence of $\lim_{t \rightarrow \infty} Q_t(\beta)$ already follows from the presence of a condensate, and that its positivity follows from the finiteness of the condensate mass. Actually, we will show a bit more. Note that weak convergence of p_t to $\rho\delta_0 + q dx$ implies that

$$\lim_{\varepsilon \downarrow 0} \lim_{t \rightarrow \infty} \int_0^\varepsilon p_t(x) dx = \rho. \quad (4.7)$$

Proposition 4.3.

(i): Assume that (1)-(3) from Proposition 4.1 hold, and that the initial condition p_0 is given by (4.3). Define $Q_t(\beta)$ as in (4.4), with $\beta = \min\{\alpha, \alpha_0\}$. Then the following two statements are equivalent:

- (a) $\lim_{t \rightarrow \infty} Q_t(\beta)$ exists and is finite.
- (b) (p_t) exhibits condensation, i.e. (4.7) holds with $\rho > 0$.

(ii): If (a) holds with $\lim_{t \rightarrow \infty} Q_t(\beta) = 0$, and if there exist $\delta > 0$ and $T > 0$ so that $p_t(x) \geq 0$ for all $x < \delta$ and all $t > T$, then we have $\rho = \infty$ in (b).

(iii): If (a) holds with $\lim_{t \rightarrow \infty} Q_t(\beta) > 0$, then for $0 < \varepsilon < \delta$,

$$\lim_{t \rightarrow \infty} \int_0^\varepsilon p_t(x) dx = \int_0^\infty \kappa(x) dx + \int_0^\varepsilon \frac{x^{\alpha-1} c_\infty(x)}{\gamma_\infty(x)} dx, \quad (4.8)$$

where $\kappa(x)$ stands for the right hand side of (4.6). Moreover,

$$\rho = \lim_{K \rightarrow \infty} \lim_{t \rightarrow \infty} \int_0^{K/t} p_t(x) dx = \int_0^\infty \kappa(x) dx. \quad (4.9)$$

Proof. We start by proving parts (ii) and (iii), which also shows that (a) implies (b) in part (i). For part (ii), note that the proof of Proposition 4.2 actually shows that

$$\lim_{t \rightarrow \infty} \frac{Q_t(\beta)}{t} p_t\left(\frac{x}{t}\right) = e^{-\gamma_\infty(0)x} \left(1_{\{\beta=\alpha\}} x^\alpha \int_0^\infty \frac{Q_s(\beta)}{(s+1)^\beta} c_s(0) ds + 1_{\{\beta=\alpha_0\}} x^{\alpha_0} \eta(0) \right)$$

under the condition that $(Q_s(\beta))$ is bounded. If $\lim_{t \rightarrow \infty} Q_t(\beta) = 0$, this implies that $\lim_{t \rightarrow \infty} \frac{1}{t} p_t(x/t) = \infty$ for all $x > 0$, and Fatou's lemma together with our positivity assumption gives

$$\liminf_{t \rightarrow \infty} \int_0^{\delta/t} p_t(x) dx = \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^\delta p_t(x/t) dx \geq \int_0^\delta \liminf_{t \rightarrow \infty} \frac{1}{t} p_t(x/t) dx = \infty.$$

This shows (ii). Now assume that $Q_\infty > 0$.

Then the rightmost equality in (4.9) is proved by observing that

$$\int_0^{K/t} p_t(x) dx = \frac{1}{t} \int_0^K p_t(x/t) dx \rightarrow \int_0^K \kappa(x) dx$$

as $t \rightarrow \infty$, by Proposition 4.2, and then taking $K \rightarrow \infty$. Let us now write

$$\tilde{p}_t(x) := W_t e^{-tx\bar{\gamma}_{0,t}(x)} p_0(x)$$

for the second term in (4.2). Then for t large enough, we have that

$$\int_{K/t}^{\varepsilon} \tilde{p}_t(x) dx = \left(\frac{t+1}{t}\right)^\beta t^{\beta-\alpha_0} Q_t(\beta)^{-1} \int_K^{t\varepsilon} e^{-x\bar{\gamma}_{0,t}(x/t)} x^{\alpha_0} \eta(x/t) dx.$$

Since $\bar{\gamma}_{0,t}(x/t)$ converges uniformly for $x \in [0, t\varepsilon]$ to $\gamma_\infty(x/t)$ (provided $\varepsilon < \delta$), and since $\gamma_\infty(x) > \gamma_- > 0$ for $x < \delta$, we find that

$$\lim_{t \rightarrow \infty} \int_K^{t\varepsilon} e^{-x\bar{\gamma}_{0,t}(x/t)} x^{\alpha_0} \eta(x/t) dx \leq Q_\infty^{-1} \int_K^\infty e^{-x\gamma_-/2} x^{\alpha_0} \sup_{z \leq \varepsilon} |\eta(z)| dx \rightarrow 0$$

as $K \rightarrow \infty$. We have shown

$$\lim_{t \rightarrow \infty} \int_0^\varepsilon \tilde{p}_t(x) dx = \lim_{K \rightarrow \infty} \lim_{t \rightarrow \infty} \int_0^{K/t} \tilde{p}_t(x) dx. \quad (4.10)$$

Now we turn to the first term of (4.2). We have

$$\begin{aligned} & \int_0^t \int_0^\varepsilon \frac{W_t}{W_s} x^\alpha c_s(x) e^{-(t-s)x\bar{\gamma}_{s,t}(x)} dx ds \\ &= \frac{1}{Q_t(\beta)} \int_0^{t/2} \int_0^{(t-s)\varepsilon} \frac{(t+1)^{1+\beta}}{(t-s)^{1+\alpha}} \frac{Q_s(\beta)}{(s+1)^{1+\beta}} z^\alpha c_s\left(\frac{z}{t-s}\right) e^{-z\bar{\gamma}_{s,t}\left(\frac{z}{t-s}\right)} dz ds \\ & \quad + \int_0^{t/2} \int_0^\varepsilon \frac{Q_{t-s}(\beta)}{Q_t(\beta)} \frac{(t+1)^{1+\alpha}}{(t-s+1)^{1+\alpha}} x^\alpha c_{t-s}(x) e^{-sx\bar{\gamma}_{t-s,t}(x)} dx ds. \end{aligned} \quad (4.11)$$

For the first integral in (4.11), we dominate the integrand uniformly by a constant multiple of the integrable function

$$\frac{1}{(s+1)^{1+\beta}} z^\alpha e^{-\gamma_- z}.$$

(Note that $\beta \leq \alpha$). If $\beta < \alpha$, then this term converges to zero. In the other case, we apply dominated convergence. In total, we obtain that the first term of (4.11) converges to

$$1_{\{\beta=\alpha\}} \frac{1}{Q_\infty(\beta)} \int_0^\infty \int_0^\infty \frac{Q_s(\beta)}{(s+1)^{1+\alpha}} z^\alpha c_s(0) e^{-z\gamma_\infty(0)} dz ds.$$

Since we assumed $Q_t(\beta)$ to converge to a strictly positive limit, the integrand of the second integral in (4.11) is bounded by a constant multiple of the integrable function

$$1_{\{x \leq \varepsilon\}} x^\alpha e^{-\gamma_- sx}.$$

It therefore converges, by dominated convergence, to

$$\int_0^\infty \int_0^\varepsilon x^\alpha c_\infty(x) e^{-sx\gamma_\infty(x)} dx ds = \int_0^\varepsilon \frac{x^{\alpha-1} c_\infty(x)}{\gamma_\infty(x)} dx.$$

If we put this together with (4.10), we have completed the proof of (4.8). The first claimed equality of (4.9) now follows by letting $\varepsilon \downarrow 0$ in (4.11).

Now we prove that (b) implies (a) in (i). Let $\mu_t(\varepsilon) := \int_0^\varepsilon p_t(x) dx$. Then, (4.2) and the definition of $Q_t(\beta)$ give

$$\begin{aligned} Q_t(\beta)\mu_t(\varepsilon) &= \int_0^t ds \int_0^\varepsilon dx (t+1)^{1+\beta} x^\alpha c_s(x) \frac{Q_s(\beta)}{(s+1)^{1+\beta}} e^{-(t-s)x\bar{\gamma}_{s,t}(x)} \\ &\quad + \int_0^\varepsilon dx (t+1)^{1+\beta} e^{-tx\bar{\gamma}_{0,t}(x)} x^{\alpha_0} \eta(x) \\ &= \int_0^t ds Q_s \frac{(t+1)^{1+\beta}}{(s+1)^{1+\beta}(t-s)^{1+\alpha}} \int_0^{\varepsilon(t-s)} dz z^\alpha c_s\left(\frac{z}{t-s}\right) e^{-z\bar{\gamma}_{s,t}(z/(t-s))} + \\ &\quad + \int_0^{t\varepsilon} dz \frac{(t+1)^{1+\beta}}{t^{1+\alpha_0}} z^{\alpha_0} e^{-z\bar{\gamma}_{0,t}(z/t)} \eta(z/t). \end{aligned} \quad (4.12)$$

Let now $\varepsilon < \delta$, where δ is as in Proposition 4.1. Let us also assume that t' is sufficiently large so that $\bar{\gamma}_{0,t'}(x) \geq \gamma_-/2$ for all $0 \leq x \leq \delta$ and all $t > t'$. This is possible since we assumed that $\bar{\gamma}_{0,t}$ converges uniformly to $\gamma_\infty > \gamma_-$ on $[0, \delta]$. Let us furthermore write $\bar{\eta} := \sup_{0 \leq x \leq \delta} \eta(x)$. Since $\beta \leq \alpha_0$, the second line of (4.12) is then bounded by

$$2\bar{\eta} \int_0^\infty dz z^{\alpha_0} e^{-z\gamma_-/2} =: D_1 < \infty, \quad (4.13)$$

for all t sufficiently large.

For the first line of (4.12), we fix $q > 1$, and for $t > 2q$ we divide the integration range $[0, t]$ of the first integral into $[0, q] \cup [q, t-q] \cup [t-q, t]$. We write

$$M_t := \sup_{s \leq t} Q_t(\beta), \quad \bar{c} := \sup\{c_s(x) : 0 \leq x \leq \delta, 0 \leq s < \infty\}.$$

Note that $\beta \leq \alpha$ and thus $(t-s)^{-1-\alpha} \leq (t-s)^{-1-\beta}$ when $s \leq t-q$, $q > 1$ and $t > 2q$. Thus the integral over $[0, q]$ is bounded by

$$M_q \bar{c} \int_0^q ds \frac{(t+1)^{1+\beta}}{(q+1)^{1+\beta}(t-q)^{1+\beta}} \int_0^\infty dz z^\alpha e^{-z\gamma_-/2} =: M_q C_1(q), \quad (4.14)$$

for all t sufficiently large. An elementary estimate shows that

$$\int_q^{t-q} \frac{(t+1)^{1+\beta}}{(s+1)^{1+\beta}(t-s)^{1+\beta}} ds \leq 2 \frac{(t+1)^{1+\beta}}{(t/2)^{1+\beta}} \int_q^{t/2} s^{-(1+\beta)} ds \leq 2^{5+2\beta} \beta^{-1} q^{-\beta},$$

and thus the integral from q to $t-q$ is bounded by

$$M_t \bar{c} \int_0^\infty dz z^\alpha e^{-z\gamma_-/2} \int_q^{t-q} ds \frac{(t+1)^{1+\beta}}{(s+1)^{1+\beta}(t-s)^{1+\beta}} =: M_t C_2 q^{-\beta} \quad (4.15)$$

for a suitable constant C_2 that is independent of t and q . Finally, for the integral from $q-t$ to t , we undo the change of variable that transformed x into $z = x/(t-s)$. Then this integral is bounded by

$$M_t \int_{t-q}^t ds \frac{(t+1)^{1+\beta}}{(s+1)^{1+\beta}} \bar{c} \int_0^\varepsilon x^\alpha dx \leq M_t q \frac{(t+1)^{1+\beta}}{(t-q+1)^{1+\beta}} \bar{c} \int_0^\varepsilon x^\alpha dx =: M_t C_3(q) \varepsilon^{1+\alpha}, \quad (4.16)$$

where $C_3(q)$ can be chosen independently of t . We now use that we assumed condensation. Let $\rho > 0$ be the mass of the condensate, let $\rho_1 < \rho$, and let ε_t be the smallest (unique, if $p_t > 0$) solution of the equation

$$\int_0^{\varepsilon_t} p_t(x) dx = \rho_1.$$

Then $\mu_t(\varepsilon_t) = \rho_1$ by definition, and $\lim_{t \rightarrow \infty} \varepsilon_t = 0$ by condensation. Now we choose $q = \bar{q}$ large enough so that the right hand side of (4.15) is less than $M_t \rho_1 / 3$. Then we choose \bar{t} large enough so that for all $t > \bar{t}$, we have $C_3(\bar{q}) \varepsilon_{\bar{t}}^{1+\alpha} \leq \rho_1 / 3$. Thus for these t , the right hand side of (4.16) is less than $M_t \rho_1 / 3$. Writing $D_2 = M_{\bar{q}} C_1(\bar{q})$ for the right hand side of (4.14), an plugging all the parts back into (4.12), we obtain the inequality

$$Q_t(\beta) \rho_1 \leq D_1 + D_2 + \frac{2}{3} M_t \rho_1,$$

valid for all $t > \bar{t}$. Then, for all $t > \bar{t}$, we get

$$M_t \leq M_{\bar{t}} + \sup_{\bar{t} \leq s \leq t} Q_s(\beta) \leq M_{\bar{t}} + \frac{D_1 + D_2}{\rho_1} + \frac{2}{3} \sup_{\bar{t} \leq s \leq t} M_s = M_{\bar{t}} + \frac{D_1 + D_2}{\rho_1} + \frac{2}{3} M_t.$$

We conclude that $M_t \leq 3(M_{\bar{t}} + (D_1 + D_2)/\rho_1)$ and thus $t \mapsto Q_t(\beta)$ is bounded.

Next we show convergence of $(Q_t(\beta))$. We have

$$\begin{aligned} Q_t(\beta) \mu_t(\varepsilon) &= \int_0^{t/2} \int_0^{(t-s)\varepsilon} Q_s(\beta) \left(\frac{t+1}{s+1}\right)^{1+\beta} \frac{1}{(t-s)^{1+\alpha}} z^\alpha c_s\left(\frac{z}{t-s}\right) e^{-z\bar{\gamma}_{s,t}\left(\frac{z}{t-s}\right)} dz ds \\ &\quad + \int_0^{t/2} \int_0^\varepsilon Q_{t-s}(\beta) \left(\frac{t+1}{t-s+1}\right)^{1+\alpha} x^\alpha c_{t-s}(x) e^{-sx\bar{\gamma}_{t-s,t}(x)} dx ds \\ &\quad + \int_0^{t\varepsilon} dz \frac{(t+1)^{1+\beta}}{t^{1+\alpha_0}} z^\alpha e^{-z\bar{\gamma}_{0,t}(z/t)} \eta(z/t). \end{aligned} \quad (4.17)$$

For the first integral on the right hand side above, we dominate the integrand uniformly by a constant multiple of

$$\frac{1}{(s+1)^{1+\alpha}} z^\alpha e^{-\gamma_- z}$$

using that $\bar{\gamma}_{s,t}(x) > \gamma_-$ for all $x \in [0, \varepsilon]$ as long as t is sufficiently large. Hence this integral converges to

$$\int_0^\infty \int_0^\infty \frac{Q_s(\beta)}{(s+1)^{1+\alpha}} z^\alpha c_s(0) e^{-z\gamma_\infty(0)} dz ds = \int_0^\infty \int_0^\infty W_s^{-1} z^\alpha c_s(0) e^{-z\gamma_\infty(0)} dz ds$$

if $\alpha = \beta$, and to zero if $\alpha > \beta$. Further the second term in (4.17) is for $t \geq t_0$ bounded by

$$U(\varepsilon) = C_1 C_4 2^{1+\alpha} \int_0^\infty \int_0^\varepsilon x^\alpha e^{-sx\bar{\gamma}_-(x)} dx ds$$

which converges to zero as $\varepsilon \downarrow 0$. By similar arguments, the third term converges to

$$\int_0^\infty z^\alpha e^{-z\gamma_\infty(0)} \eta(0) dz$$

if $\beta = \alpha_0$, and to zero if $\beta < \alpha_0$. Defining

$$R := 1_{\{\beta=\alpha\}} \int_0^\infty \int_0^\infty W_s^{-1} z^\alpha c_s(0) e^{-z\gamma_\infty(0)} dz ds + 1_{\{\beta=\alpha_0\}} \int_0^\infty z^\alpha e^{-z\gamma_\infty(0)} \eta(0) dz,$$

we see from (4.17) and the assumption that $\lim_{\varepsilon \downarrow 0} \liminf_{t \rightarrow \infty} \mu_t(\varepsilon) = \rho > 0$ that

$$\limsup_{t \rightarrow \infty} Q_t(\beta) \leq \lim_{\varepsilon \downarrow 0} \frac{1}{\liminf_{t \rightarrow \infty} \mu_t(\varepsilon)} \left(R + U(\varepsilon) \right) = \frac{1}{\rho} R.$$

Analogously, using that $\lim_{\varepsilon \downarrow 0} \limsup_{t \rightarrow \infty} \mu_t(\varepsilon) = \rho > 0$ we get that

$$\liminf_{t \rightarrow \infty} Q_t(\beta) \geq \frac{1}{\rho} R.$$

Hence $(Q_t(\beta))$ converges to R/ρ . \square

The proof of Theorem 2.2 is thus finished. Under its assumptions, Proposition 4.1 guarantees that the solution to (2.1) fulfills equation (4.1) with suitable properties of b , c and γ . Since a finite, nonnegative condensation at $x = 0$ is assumed, Proposition 4.3 guarantees that Q_∞ exists, is finite and strictly positive. Then, Proposition 4.2 shows the claim of the proof. Note that equation (4.8) confirms that the shape of the bulk density needs to be

$$q(x) = \frac{x^{\alpha-1}c_\infty(x)}{\gamma_\infty(x)} = \frac{x^\alpha \mathbf{C}[p_\infty](x)}{\mathbf{B}[p_\infty](x)},$$

for $x > 0$, a fact that already follows from the stationarity of p_∞ for equation (2.1). Also, equation (4.9) shows that all of the condensate forms on a scale of $1/t$ as $t \rightarrow \infty$.

5. APPENDIX: PROOF OF PROPOSITION 2.1

Fix $t > 0$. Then by our assumptions on K_b ,

$$\int_0^\infty \sup_{x \in [0, \delta]} |\partial_x K_b(x, y)| |p_t(y) - q(y)| dy < \infty.$$

Thus Lebesgue's theorem allows us to differentiate under the integral sign and obtain

$$\begin{aligned} \partial_x(\mathbf{B}[p_t](x) - \mathbf{B}[p_\infty](x)) &= \int_0^\infty \partial_x K_b(x, y)(p_t(y) - q(y)) dy - \rho \partial_x K_b(x, 0) \\ &= \int_0^\infty \partial_x K_b(x, y)(p_t(dx) - p_\infty(dx)). \end{aligned}$$

Furthermore,

$$\mathbf{C}[p_t](x) - \mathbf{C}[p_\infty](x) = \int_0^\infty K_c(x, y)(p_t(dx) - p_\infty(dx)),$$

with a similar equation for $\mathbf{B}[p_t](x) - \mathbf{B}[p_\infty](x)$. Thus the claim will be shown once we prove the following lemma:

Lemma 5.1. *Let (p_t) and $p_\infty = \rho\delta_0 + q(x)dx$ fulfill the assumptions of Proposition 2.1. Then we have*

$$\lim_{t \rightarrow \infty} \sup_{0 \leq x \leq \delta} \left| \int_0^\infty h(x, y)(p_t(dy) - p_\infty(dy)) \right| = 0$$

for all $\delta > 0$ and all $h \in C_b([0, \delta] \times \mathbb{R}_0^+)$.

Proof. First we rewrite

$$\begin{aligned} \int_0^\infty h(x, y)(p_t(y) - q(y)) dy - \rho h(x, 0) &= \int_0^\infty (h(x, y) - h(x, 0))(p_t(y) - q(y)) dy \\ &\quad + h(x, 0) \left(\int_0^\infty (p_t(y) - q(y)) dy - \rho \right) \end{aligned} \quad (5.1)$$

By the weak convergence of $p_t dx$ to $q dx + \rho\delta_0$, the second line of (5.1) converges to zero uniformly in $x \in [0, \delta]$. For the first line, we fix $\varepsilon > 0$. Since h is continuous, it is uniformly continuous on compact intervals. Therefore there exists $\tilde{\delta} > 0$ so that for all $x \in [0, \delta]$ and all $y \in [0, \tilde{\delta}]$, we have $|h(x, y) - h(x, 0)| < \varepsilon$. Thus

$$\left| \int_0^{\tilde{\delta}} (h(x, y) - h(x, 0))(p_t(y) - q(y)) dy \right| \leq \varepsilon \int_0^{\tilde{\delta}} |p_t(y)| + |q(y)| dy. \quad (5.2)$$

By decreasing $\tilde{\delta}$ if necessary and using condition (2.5), we find that the latter integral is bounded by a constant C , and hence (5.2) is bounded by $C\varepsilon$ uniformly in $x \in [0, \delta]$. Note that for sufficiently large t this would follow already from weak convergence of p_t if we had assumed that p_t is nonnegative. For the remaining part of the first line of (5.1), we estimate

$$\int_{\tilde{\delta}}^{\infty} (h(x, y) - h(x, 0))(p_t(y) - q(y)) \, dy \leq 2\|h\|_{\infty} \int_{\tilde{\delta}}^{\infty} |p_t(y) - q(y)| \, dy. \quad (5.3)$$

By assumption (2.4), this integral tends to zero uniformly in x . Putting (5.2) and (5.3) together, we finally get that

$$\limsup_{t \rightarrow \infty} \sup_{0 \leq x \leq \delta} \left| \int_0^{\infty} (h(x, y) - h(x, 0))(p_t(y) - q(y)) \, dy \right| \leq C\varepsilon$$

and the claim follows since $\varepsilon > 0$ was arbitrary. \square

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