The size of the giant in inhomogeneous random graphs of preferential attachment type

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Abstract. For the inhomogeneous random graph with kernel of preferential attachment type and degree distribution with power-law exponent $\tau \in (2,3)$ we study the decay of the size of the giant component when the edge density approaches zero. It turns out that the giant component is significantly smaller than for the inhomogeneous random graph with a kernel of rank one.

Keywords: Scale-free network · preferential attachment · small giant · phase transition.

1 Introduction

The inhomogeneous random graph with kernel of preferential attachment type is a solvable model, which shares many features of more involved preferential attachment models. If set-up with a power-law exponent $\tau \in (2,3)$ the model is *robust* in the sense that there is a connected component comprising an asymptotically positive proportion of the vertices, no matter how small the edge density. However, the asymptotic proportion of vertices in this giant component decreases very quickly to zero when the edge density decreases to zero. Our result identifies the exact speed at which this happens.

Inhomogeneous random graphs cf. [3], [9, Chapter 3.2], [14], are parametrised by a symmetric, continuous kernel

$$\kappa\colon (0,1] \times (0,1] \to [0,\infty).$$

Given the kernel, the graph $\mathcal{G}_n$ has vertex set $\{1,\dots,n\}$ and any pair of distinct vertices $i,j \in \{1,\dots,n\}$ is connected by an edge, independently with probability

$$p_{i,j} := \frac{1}{n}\kappa\Big(\frac{i}{n},\frac{j}{n}\Big) \wedge 1.$$

In most preferential attachment models the probability at which a vertex arriving at time j connects to an earlier vertex i is proportional to its degree. This degree is of order $(j/i)^\gamma$ for some $0<\gamma<1$, see for example [7], and the proportionality factor therefore inverse to order $\sum_{i=1}^{j-1}(j/i)^\gamma \sim j$. Therefore in order to get edge probabilities that match those in the preferential attachment models we choose the kernel κ of preferential attachment type

$$\kappa(x,y) = \beta(x\wedge y)^{-\gamma}(x\vee y)^{\gamma-1},$$

for some density parameter $\beta>0$ and attachment strength parameter $0<\gamma<1$. It is easy to check that the graph thus constructed has a power-law degree distribution with exponent

$$\tau = 1 + \frac{1}{\gamma}.$$

Note that because the kernel is homogeneous of index -1, the connection probabilities $p_{i,j}$ do not depend on n. This reflects the dynamic nature of the preferential attachment model. When $\gamma > \frac{1}{2}$ the power-law exponent τ lies between 2 and 3, a range which is often desired when modelling scale-free networks. Precisely in this case the largest component in the graph has macroscopic size no matter how small the edge density, see Dereich et al [6], or [12]. Our result determines the asymptotic size of this component when the edge density is small.

Theorem 1. *Let $\frac{1}{2} \leq \gamma < 1$ and write $C_1(\mathcal{G}_n)$ for the number of vertices in the largest component. Then we have*

$$\lim_{n\to\infty} \frac{C_1(\mathcal{G}_n)}{n} = \exp\left(-(1-\gamma)\frac{1+o(1)}{\beta}\right),$$

where the limit is taken in probability, the limiting quantity is deterministic and $o(1)$ refers to the asymptotics $\beta \downarrow 0$.

Remarks:

(i) It should be noted that the largest component is surprisingly small. Cohen et al [4] predict a much larger asymptotic proportion of order $\beta^{1/(\tau-2)}$ for the size of the largest component in scale-free networks with power-law exponent $\tau \in (2,3)$. This prediction is accurate for the inhomogeneous random graph with the same power-law exponent based on the rank-one kernel , $\kappa(x,y) = \beta(xy)^{-\gamma}$, see [3]. The significant difference in the size of the giant after percolation with a small retention parameter between these models may be used in practice to test whether network data credibly originates from a preferential attachment scheme.

(ii) The case $\gamma = \frac{1}{2}$ is treated by Bollobas et al in [3]. In [13] Riordan describes the observation of the surprisingly small components as *small giant phenomenon.* Note that in the case $\gamma = \frac{1}{2}$ preferential attachment and rank-one kernel coincide.

(iii) Eckhoff et al [8] have an analogous result for the regime $0 < \gamma < \frac{1}{2}$ showing a decay of order $e^{-c/\sqrt{\beta-\beta_c}}$ with $c = \frac{\pi}{\sqrt{4-8\gamma}}$ when $\beta \downarrow \beta_c > 0$. In that paper a more involved model is considered, but the result can be adapted to our model, see [10] for details.

(iv) The techniques used in [8] and [3] rely on local neighbourhood approximation of the network and analysis of the survival probability of the approximating tree. This method is not suited to our case as the approximating tree has a superexponential growth and is not easy to handle. Instead, we give a constructive proof for the upper bound based on path counting techniques, relying heavily on the fact that these networks are ultra-small, that is typical distances between vertices are only of the order $\log\log n$. For the lower bound we cut-off the small index vertices from our graph, which allows us to use classical theory of inhomogeneous random graphs, as developed in [3].

2 Proof of Theorem 1.

As the case $\gamma = \frac{1}{2}$ is covered by Riordan in [13, (4.1)] we assume from now on that $\gamma \in (\frac{1}{2}, 1)$. We start with the upper bound, which we prove by a path counting argument. In Section 3 we show that there exists a *core* of highly connected vertices in the network, with bounded diameter independent of n. The core consists of the vertices with index less or equal to $m := m(n) := \sqrt{n}\,(\log n)^{-\alpha}$, for $\alpha = (4-2\gamma)^{-1}$, which satisfy a mild regularity condition, see Definition 1 and the lemmas thereafter. With high probability any giant component contains the core and thus intersects the set $\{1, \ldots, m\}$. Hence we can upper bound the number of vertices in the largest component by m plus the number of (self-avoiding) paths which connect a non-core vertex $i > m$ with the set $\{1, \ldots, m\}$. To count these paths we use two key observations. The first is a bound on the maximum length of a shortest path. Recall the definition of the graph distance of two vertices x, y, namely

$$d_n(x,y) := \min\left\{N \colon \exists x = x_0, ..., x_N = y \in \mathcal{G}_n, \text{ such that } x_{i-1} \leftrightarrow x_i \forall i \in \{1, ..., N\}\right\}$$

Obviously any path can at most be of length n before it loops into itself, but actual shortest paths are much shorter. Indeed the typical shortest paths in the giant component are no longer than order $\log\log(n)$.

Proposition 1. *Denote by $\mathcal{C}_1(\mathcal{G}_n)$ the largest component in $\mathcal{G}_n$. Then there exists a constant $B > 0$ such that, in probability,*

$$\frac{1}{n^2}\#\left\{(x,y) \in \mathcal{C}_1(\mathcal{G}_n)^2 \colon d_n(x,y) \geq 4\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + B\right\} \to 0\,.$$

This result was already shown by Mönch for various preferential attachment models in [11] and the proof for the inhomogeneous random graph of preferential attachment type is deferred to Section 3. In the proof we show the following subresult, see Lemmas 6 and 7,

$$\frac{1}{n}\#\Big\{x \in \mathcal{C}_1(\mathcal{G}_n)\colon d_n(x, \mathsf{core}_n) \geq 2\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + B\Big\} \to 0 \, .$$

The second observation is that the number of paths connecting to the core is, in expectation, dominated by the paths, which connect the non-core vertices quickly to a powerful vertex with a small index. To count them, define a decreasing cutoff-sequence $(t_l)_{l\in\mathbb{N}_0} \subset (0,1)$ and let $X_l^{(1)}$ be the number of paths $x_0x_1\cdots x_{l-1}x_l$ of length l, which satisfy the condition $x_k > t_k n$ for all $0 \leq k < l$ and $x_l \leq t_l n$. These are the paths, which hit an early vertex rather fast. The remaining paths stay above the threshold at all times until they arrive at the core. Let $X_l^{(2)}$ be the number of paths of length l, such that $x_k > t_k n$ for all $0 \leq k \leq l$ and $x_l \leq m$. Let $l^* := l^*(n) := \min\{l \, : \, t_l n \leq m(n)\}$ and set

$$L := L(n) := 2\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + 2\,C + K \, , \tag{1}$$

where C, K are the constants from Section 3, i.e. $C = \frac{\log(\beta^2)}{\log(\frac{\gamma}{1-\gamma})}$ and K depends neither on n nor on β. Then, by our observations, we can upper bound the number of vertices in the largest component by

$$C_1(\mathcal{G}_n) \leq t_0 n + \sum_{l=1}^{L} X_l^{(1)} + \sum_{l=l^*}^{L} X_l^{(2)} + o(n), \tag{2}$$

in probability, where the first term includes vertices which are already in the core. Indeed, with high probability almost all vertices in the largest component can be connected to the core by a shortest path of length no longer than L by Lemmas 6 and 7. Now starting with the first sum, we investigate its expectation

$$\mathbb{E}[X_l^{(1)}] \leq \sum_{0<x_l\leq t_l n} \; \sum_{t_{l-1}n<x_{l-1}\leq n} \cdots \sum_{t_0 n<x_0\leq n} \prod_{i=0}^{l-1} \beta(x_i \vee x_{i+1})^{\gamma-1}(x_i \wedge x_{i+1})^{-\gamma} \, .$$

Note that in the above we would actually get an equality if we restrict the summation variables to be distinct. We define, for all $k \in \{1,\dots,l\}$,

$$\mu_k(x_k) := \sum_{t_{k-1}n<x_{k-1}\leq n} \cdots \sum_{t_0 n<x_0\leq n} \prod_{i=0}^{k-1} \underbrace{\beta(x_i \vee x_{i+1})^{\gamma-1}(x_i \wedge x_{i+1})^{-\gamma}}_{=:p(x_i,x_{i+1})},$$

which we can thus express recursively

$$\mu_{k+1}(x) = \sum_{t_k n<y\leq n} \mu_k(y)p(x,y) \quad \forall 1 \leq k < l \, . \tag{3}$$

Moreover from the definition of μ_k immediately follows that

$$\mathbb{E}[X_l^{(1)}] \leq \sum_{0<x_l\leq t_l n} \mu_l(x_l) \, . \tag{4}$$

Going forward we use the recursive representation of μ_k to obtain an upper bound. The following statement was already shown in [7], compare with Lemma 1 there.

Lemma 1. *Let $\mu : \{1,\dots,n\} \to [0,\infty)$ be a function satisfying*

$$\mu(x) \leq \mathbb{1}_{\{x\geq \bar{t}n\}}\psi x^{\gamma-1} + \phi x^{-\gamma}$$

for all $x \in \{1, \dots, n\}$, for some constants $\psi, \phi > 0$ and some cutoff $\bar{t} \in (0,1)$. Then, for any $t \le \bar{t}$,

$$\sum_{tn<y\le n} \mu(y)p(x,y) \le \mathbb{1}_{\{x>tn\}}\beta\left(\frac{\phi}{2\gamma-1}(tn)^{1-2\gamma} + \psi\log\left(\frac{1}{t}\right)\right)x^{\gamma-1} + \beta\left(\phi\log\left(\frac{1}{t}\right) + \frac{\psi}{2\gamma-1}n^{2\gamma-1}\right)x^{-\gamma}.$$

Proof. We have

$$\begin{aligned}
\sum_{tn<y\le n} \mu(y)p(x,y) &= \mathbb{1}_{\{x>tn\}} \sum_{tn<y<x} \mu(y)p(x,y) + \sum_{y=x}^{n} \mu(y)p(x,y) \\
&= \mathbb{1}_{\{x>tn\}} \sum_{tn<y<x} \mu(y)\beta y^{-\gamma}x^{\gamma-1} + \sum_{y=x\vee tn}^{n} \mu(y)\beta y^{\gamma-1}x^{-\gamma} \\
&\le \mathbb{1}_{\{x>tn\}}\beta \sum_{tn<y<x} (\psi y^{-1} + \phi y^{-2\gamma})x^{\gamma-1} + \beta\sum_{y=x\vee tn}^{n}(\psi y^{2(\gamma-1)} + \phi y^{-1})x^{-\gamma} \\
&\le \mathbb{1}_{\{x>tn\}}\beta\left(\psi\log\left(\tfrac{x}{tn}\right) + \tfrac{\phi}{2\gamma-1}(tn)^{1-2\gamma}\right)x^{\gamma-1} + \beta\left(\tfrac{\psi}{2\gamma-1}n^{2\gamma-1} + \phi\log\left(\tfrac{n}{tn}\right)\right)x^{-\gamma} \\
&\le \mathbb{1}_{\{x>tn\}}\beta\left(\psi\log\left(\tfrac{1}{t}\right) + \tfrac{\phi}{2\gamma-1}(tn)^{1-2\gamma}\right)x^{\gamma-1} + \beta\left(\tfrac{\psi}{2\gamma-1}n^{2\gamma-1} + \phi\log\left(\tfrac{1}{t}\right)\right)x^{-\gamma}.
\end{aligned}$$

If β is sufficiently small (e.g. $\beta \le 2\gamma - 1$) we get the following corollary,

$$\sum_{tn<y\le n} \mu(y)p(x,y) \le \mathbb{1}_{\{x>tn\}}\left(\phi(tn)^{1-2\gamma} + \psi\beta\log\left(\tfrac{1}{t}\right)\right)x^{\gamma-1} + \left(\phi\beta\log\left(\tfrac{1}{t}\right) + \psi n^{2\gamma-1}\right)x^{-\gamma}.$$

Going forward we apply Lemma 1 inductively with $\bar{t} = t_{k-1}$ and $t = t_k$ in every step k, however to this end we first need the starting value

$$\begin{aligned}
\mu_1(x_1) &= \sum_{t_0 n<x_0\le n} \beta(x_0 \vee x_1)^{\gamma-1}(x_0 \wedge x_1)^{-\gamma} \\
&\le \beta \sum_{t_0 n<x_0\le n} \left(\mathbb{1}_{\{x_1>t_0 n\}}x_1^{\gamma-1}x_0^{-\gamma} + x_0^{\gamma-1}x_1^{-\gamma}\right) \\
&\le \beta\left(\mathbb{1}_{\{x_1>t_0 n\}}\left[\tfrac{1}{1-\gamma}x_0^{1-\gamma}\right]_{x_0=t_0 n}^{n} x_1^{\gamma-1} + \left[\tfrac{1}{\gamma}x_0^{\gamma}\right]_{x_0=t_0 n}^{n} x_1^{-\gamma}\right) \\
&\le \mathbb{1}_{\{x_1>t_0 n\}}\frac{\beta n^{1-\gamma}}{1-\gamma}x_1^{\gamma-1} + \frac{\beta n^{\gamma}}{\gamma}x_1^{-\gamma}.
\end{aligned}$$

We factor out n and let $\psi_1 = \frac{\beta n^{-\gamma}}{1-\gamma}$ and $\phi_1 = \frac{\beta n^{\gamma-1}}{\gamma}$ and define for all $k \ge 1$, by the above corollary

$$\begin{aligned}
\psi_{k+1} &= \phi_k(t_k n)^{1-2\gamma} + \psi_k\beta\log\left(\tfrac{1}{t_k}\right) \\
\phi_{k+1} &= \phi_k\beta\log\left(\tfrac{1}{t_k}\right) + \psi_k n^{2\gamma-1}
\end{aligned}$$

It will later come in handy that we define our sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k n$ is the largest integer satisfying the following relation

$$\frac{1}{1-\gamma}\phi_k(t_k n)^{1-\gamma} \le \frac{6\,t_0^{1-\gamma}}{\pi^2 k^2}\ . \tag{5}$$

One can see by definition of ϕ_k above, that $\phi_k = O(n^{\gamma-1})$ and thus t_k really only depends on n in the sense that $t_k n$ has to be an integer. Note that by definition of (ϕ_k) this is a recursive definition for $(t_k)_{k\in\mathbb{N}}$, given t_0 which we are still free to choose. Later on we also need an upper bound for the growth of t_k^{-1}, which we establish now using only the definitions of ϕ_k and t_k. From the definition of t_k follows immediately that

$$(t_k n + 1)^{1-\gamma} \geq \frac{6(1-\gamma)t_0^{1-\gamma}}{\pi^2 k^2 \phi_k},$$

and hence

$$\begin{aligned}
(t_{k+2} + \frac{1}{n})^{\gamma-1} &\leq \frac{\pi^2(k+2)^2}{6(1-\gamma)t_0}\phi_{k+2} n^{1-\gamma} \\
&\leq \frac{\pi^2(k+2)^2}{6(1-\gamma)t_0}\left(\phi_{k+1}\beta\log\left(\frac{1}{t_{k+1}}\right) + \psi_{k+1}n^{2\gamma-1}\right)n^{1-\gamma} \\
&\leq \frac{(k+2)^2}{(k+1)^2}t_{k+1}^{\gamma-1}\beta\log\left(\frac{1}{t_{k+1}}\right) + \frac{\pi^2(k+2)^2}{6(1-\gamma)t_0}\left(\phi_k(t_k n)^{1-2\gamma} + \psi_k\beta\log\left(\frac{1}{t_k}\right)\right)n^{\gamma} \\
&\leq \frac{(k+2)^2}{(k+1)^2}t_{k+1}^{\gamma-1}\log\left(\frac{1}{t_{k+1}}\right) + \frac{(k+2)^2}{k^2}t_k^{-\gamma} + \frac{\pi^2(k+2)^2}{6(1-\gamma)t_0}\phi_{k+1}\beta\log\left(\frac{1}{t_k}\right)n^{1-\gamma} \\
&\leq 2\frac{(k+2)^2}{(k+1)^2}t_{k+1}^{\gamma-1}\beta\log\left(\frac{1}{t_{k+1}}\right) + \frac{(k+2)^2}{k^2}t_k^{-\gamma}.
\end{aligned}$$

Using the above inequality we want to prove the following claim, for all $k \geq 0$ there exists a constant c, independent of β and n, such that

$$t_k^{-1} \leq c\exp\left(\frac{1}{\beta}\sqrt{\frac{\gamma}{1-\gamma}}^{k}\right). \tag{6}$$

In accordance with the above we pick $t_0 = \varepsilon e^{-1/\beta}$, where ε is just to ensure that $t_0 n$ is an integer. One can easily verify by (5) that t_1, t_2 satisfy inequality (6). For all $k \geq 1$ we have

$$\begin{aligned}
t_{k+2}^{-1} &\leq \left(2\frac{(k+2)^2}{(k+1)^2}t_{k+1}^{\gamma-1}\beta\log\left(\frac{1}{t_{k+1}}\right) + \frac{(k+2)^2}{k^2}t_k^{-\gamma}\right)^{\frac{1}{1-\gamma}} \\
&\leq \left(c_1\exp\left(\frac{1}{\beta}\sqrt{\frac{\gamma^{k+1}}{(1-\gamma)^{k-1}}}\right)\left(\sqrt{\frac{\gamma}{1-\gamma}}\right)^{k+1} + c_2\exp\left(\frac{1}{\beta}\sqrt{\frac{\gamma^{k+2}}{(1-\gamma)^{k}}}\right)\right)^{\frac{1}{1-\gamma}}
\end{aligned}$$

for some constants c_1 and c_2. Now the claim is implied by the fact that

$$\exp\left(\frac{1}{\beta}\sqrt{\frac{\gamma^{k+1}}{(1-\gamma)^{k-1}}}\right)\left(\sqrt{\frac{\gamma}{1-\gamma}}\right)^{k+1} \leq \tilde{c}\exp\left(\frac{1}{\beta}\sqrt{\frac{\gamma^{k+2}}{(1-\gamma)^{k}}}\right),$$

for a suitable constant $\tilde{c}$. We apply Lemma 1 combined with (4) and (5), and do not forget the n we factored out earlier, to get

$$\mathbb{E}[X_l^{(1)}] \leq \sum_{0<x_l\leq t_l n}\mu_l(x_l) \leq \sum_{0<x_l\leq t_l n} n\phi_l x_l^{-\gamma} \leq n\frac{1}{1-\gamma}\phi_l(t_l n)^{1-\gamma} \leq n\frac{6\,t_0^{1-\gamma}}{\pi^2 l^2}$$

Summing over all l thus gives us

$$\sum_{l\geq 1}\mathbb{E}[X_l^{(1)}] \leq n t_0^{1-\gamma} \tag{7}$$

The paths in $X^{(2)}$ stay above the threshold at all times until they reach the core. Thus for any $l \geq l^*$ we can write the expectation as

$$\begin{aligned}
\mathbb{E}[X_l^{(2)}] &\leq \sum_{0<x_l\leq m}\ \sum_{t_{l-1}n<x_{l-1}\leq n}\cdots\sum_{t_0 n<x_0\leq n}\prod_{i=0}^{l-1}\beta(x_i\vee x_{i+1})^{\gamma-1}(x_i\wedge x_{i+1})^{-\gamma} \\
&\leq \sum_{0<x_l\leq m}\mu_l(x_l)\,.
\end{aligned}$$

By the definition of L in (1) it follows from our estimation (6) that $l^* = \min\{l : t_l n \le m(n)\}$ is asymptotically equal to L, i.e. $L = l^* + c_1$ for some constant c_1 independent of n. By the bound (6) and (1) we get

$$t_L^{-1} \le c \exp\left(\frac{1}{\beta} \left(\frac{\gamma}{1-\gamma} \right)^{L/2} \right) \le c\, n^{\beta \left(\sqrt{\frac{\gamma}{1-\gamma}}\right)^K} .$$

Also recall that $m = \sqrt{n}(\log n)^{-\alpha}$, $\alpha > 0$. Hence we find a constant c_2, such that

$$\begin{aligned} \sum_{l=l^*}^{L} \mathbb{E}[X_l^{(2)}] &\le \sum_{l=l^*}^{L} \sum_{0<x_l\le m} n\phi_l x_l^{-\gamma} \le \frac{c_2}{1-\gamma} n^{1-\gamma+\frac{1}{2}(1+\gamma)} \phi_L \\ &\le \frac{c_3\, t_0^{1-\gamma}}{L^2} n^{\frac{1}{2}(1+\gamma)} t_L^{\gamma-1} \le \frac{c_3\, t_0^{1-\gamma}}{L^2} n^{\beta\, \tilde{c} + \frac{1}{2}(1+\gamma)} = o(n) . \end{aligned}$$

The second to last inequality follows by (5) for a suitable constant c_3 and we get that the term is $o(n)$ for β small enough, since $\frac{1}{2}(1+\gamma) < 1$. Thus we use the above together with (7) in (2) to obtain

$$\mathbb{E}[C_1(\mathcal{G}_n)] \le t_0 n + t_0^{1-\gamma} n + o(n)$$

as $n \to \infty$. For $\beta \to 0$ the t_0 term with the lowest exponent dominates, which in this case is $t_0^{1-\gamma}$. Hence by our choice of t_0 in (6) the upper bound in Theorem 1 follows in expectation.

Now we want to conclude from this that the statement also holds with high probability. So from now on let $X_l = X_l^{(1)} + X_l^{(2)}$ be the number of all *core paths* of length l, note that by definition these paths are not necessarily shortest paths and do not necessarily end in the core. For fixed $l \ge 1$ we want to show the concentration of X_l, to this end we will use McDiarmid's inequality in the enhanced version of [5] which we now restate. Consider a function $f\colon \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N \to \mathbb{R}$, where each $\mathcal{Y}_i$ is a probability space. Moreover, let $G \subseteq \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_N$ be in the domain of f. We call this the *good event* and say that f satisfies a *bounded difference inequality on* G if there exists constants $c_1, \ldots, c_N$ such that, for all $i \in \{1, \ldots, N\}$ and for all $(y_1, \ldots, y_N), (y'_1, \ldots, y'_N) \in G$ with $y_j = y'_j$ for all $j \ne i$,

$$|f(y_1, \ldots, y_N) - f(y'_1, \ldots, y'_N)| \le c_i .$$

For such f, for any independent random variables $Y_1, \ldots, Y_N$ on the spaces $\mathcal{Y}_1, \ldots, \mathcal{Y}_N$ and for all $\delta > \mathbb{P}(G^c) \sum_{i=1}^N c_i$, it holds that

$$\begin{aligned} &\mathbb{P}\big(|f(Y_1, \ldots, Y_N) - \mathbb{E}[f(Y_1, \ldots, Y_N) \,|\, (Y_1, \ldots, Y_N) \in G]| > \delta\big) \\ &\le 2\mathbb{P}(G^c) + 2\exp\left(-\frac{2(\delta - \mathbb{P}(G^c)\sum_{i=1}^N c_i)^2}{\sum_{i=1}^N c_i^2} \right) . \end{aligned} \tag{8}$$

First we find independent random variables $Y_{\hat{n}_l+1}^{\text{in}}, Y_{\hat{n}_l+1}^{\text{out}}, \ldots, Y_n^{\text{in}}, Y_n^{\text{out}}$ and a function f such that $X_l = f(Y_{t_l n}^{\text{in}}, Y_{t_l n}^{\text{out}}, \ldots, Y_n^{\text{in}}, Y_n^{\text{out}})$, where $\hat{n}_l = t_l n \vee m(n)$, consequently $N = 2(n - \hat{n}_l - 1)$. We choose $Y_i^{\text{in}} \in \{0,1\}^n$ to be the vector corresponding to the incoming connections of vertex i and $Y_i^{\text{out}} \in \{0,1\}^n$ to be the vector corresponding to the outgoing connections of vertex i into the set $\{1, \ldots, \hat{n}_l\}$. Moreover let $\mathcal{Y}_i^{\text{in}}, \mathcal{Y}_i^{\text{out}}$ be their canonical probability spaces. Thus f has to be the number of all core-paths as a function of $Y_{\hat{n}_l+1}^{\text{in}}, Y_{\hat{n}_l+1}^{\text{out}}, \ldots, Y_n^{\text{in}}, Y_n^{\text{out}}$, i.e. let

$$I_n := \{(i_0, \ldots, i_l) \in \{1, \ldots, n\}^{l+1} : i_k > t_k n \,\forall k < l,\, i_l \le \hat{n}_l,\, i_h \ne i_j \,\forall h \ne j\}$$

be the set of all core paths in the full graph and

$$f(Y_{\hat{n}_l+1}^{\text{in}}, Y_{\hat{n}_l+1}^{\text{out}}, \ldots, Y_n^{\text{in}}, Y_n^{\text{out}}) := \sum_{(i_0,\ldots,i_l)\in I_n} Y_{i_{l-1}}^{\text{out}}(i_l) \prod_{j=0}^{l-2} \left(Y_{i_j}^{\text{in}}(i_{j+1}) \vee Y_{i_{j+1}}^{\text{in}}(i_j)\right) .$$

We want our good event to be the event that none of the vertices in our graph has untypically many connections, what is untypical for a vertex depends on its index. Therefore we group the vertices into disjoint *layers* $N^{(0)} := \{t_0 n, \dots, n\}$, $N^{(k)} := \{t_k n, \dots, t_{k-1}n - 1\}$ for all $k = 1, \dots, L-1$ and $N^{(L)} := \{1, \dots, m(n)\}$. Recall L in (1), moreover let $\pi_n \colon \{1, \dots, n\} \to \{0, \dots, L\}$ be the function which maps a vertex onto its layer. Then we define our good event to be

$$G := \big\{ \forall i \in \{\hat{n}_l + 1, \dots, n\} \, : \, |Y_i^{\text{in}}| + |Y_i^{\text{out}}| + \sum_{j=\hat{n}_l+1}^{i-1} Y_j(i) \le a_{\pi_n(i)}(n) \big\}$$

for some constants $a_0(n), \dots, a_L(n)$ depending on n and $|Y_i| := \sum_{j=1}^n Y_i(j)$. We can write $|Y_i^{\text{in}}| = \sum_{j=i+1}^n U_j$ for independent Bernoulli random variables with parameters $p_j = \beta i^{-\gamma} j^{\gamma-1}$ and $|Y_i^{\text{out}}| + \sum_j Y_j(i) = \sum_{j=1}^{t_k n-1} U_j$ with parameter $p_j = \beta j^{-\gamma} i^{\gamma-1}$. We write $q_j = 1 - p_j$ for all j, also note that for all $j \ge t_l n$ we have $p_j \le \beta i^{-1} =: p$ and $q_j \le 1 - \beta n^{\gamma-1} i^{-\gamma} =: q$. Thus, by the Markov inequality,

$$\begin{aligned}\mathbb{P}\big(|Y_{t_k n}^{\text{in}}| > \tfrac{a_k(n)}{2}\big) &\le \prod_{j=t_k n+1}^{n} \mathbb{E}\left[e^{U_j}\right] e^{-a_k(n)} \le (q + pe)^n e^{-a_k(n)/2} \\ &\le \left(1 + \frac{\beta t_k^{-1} e - \beta}{n}\right)^n e^{-a_k(n)/2} \, ,\end{aligned}$$

which is $o(\frac{1}{n})$ for $a_k(n)/2 := \beta t_k^{-1} e + \log(n)^2$. The same holds for the outgoing connections as

$$\begin{aligned}\mathbb{P}\big(|Y_{t_k n}^{\text{out}}| + \sum_{j=\hat{n}_l+1}^{t_k n-1} Y_j(t_k n) > \tfrac{a_k(n)}{2}\big) \le& \prod_{j=1}^{t_k n} \mathbb{E}\left[e^{U_j}\right] e^{-a_k(n)} \\ \le& \prod_{j=1}^{t_k n} \left(1 + \frac{\beta t_k^{\gamma-1} e - \beta}{n^{1-\gamma}} (t_k n)^{1-\gamma} j^{-1}\right) e^{-a_k(n)/2} \\ \le& (t_k n + 1) e^{-a_k(n)/2} \, ,\end{aligned}$$

where we used that $\beta(t_k^{\gamma-1} e - 1)(\frac{t_k n}{n})^{1-\gamma} \le 1$ for β small enough and that $\prod_{j=1}^n (1 + j^{-1}) = n + 1$. Hence we get for the probability that our good event does not occur

$$\begin{aligned}\mathbb{P}(G^c) =& \mathbb{P}\big(\exists i \, : \, |Y_i^{\text{in}}| + |Y_i^{\text{out}}| + \sum_{j=\hat{n}_l+1}^{i-1} Y_j(i) > a_{\pi_n(i)}(n)\big) \\ \le& \sum_{k=0}^{l} (t_{k-1} - t_k) n \left(\mathbb{P}\big(|Y_{t_k n}^{\text{in}}| > a_k(n)/2\big) + \mathbb{P}\big(|Y_{t_k n}^{\text{out}}| + \sum_{j=\hat{n}_l+1}^{t_k n-1} Y_j(t_k n) > a_k(n)/2\big) \right) \\ \le& C n^2 e^{-\log(n)^2} \, .\end{aligned} \tag{9}$$

Furthermore we have for all $i \in \{1, \dots, N\}$ and for all $(y_1, \dots, y_N), (y'_1, \dots, y'_N) \in G$ with $y_j = y'_j$ for all $j \ne i$

$$|f(y_1, \dots, y_N) - f(y'_1, \dots, y'_N)| \le l \prod_{j=0}^{l} a_j(n) =: c_i \, .$$

Indeed, since l is fixed, the maximum difference is achieved by setting $y'_i = 0$ and choosing y_i such that $|y_i| = a_{\pi_n(i)}(n)$ or vice versa. Thus the difference above is bounded by the number of core paths in the graph corresponding to $(y_1, \dots, y_N)$, which involve vertex i. For any path we have at most l places for i and since we work on G any vertex j has at most $a_{\pi_n(j)}(n)$ connections and hence the claim follows.

As c_i does not depend on i, we have for the sum

$$2\sum_{i=t_l n}^{n}\left(l\prod_{j=0}^{l}a_j(n)\right)^2 \le 2nl^2\prod_{i=0}^{l}\left(2\beta t_i^{-1}e+2\log(n)^2\right)^2$$
$$\le 2nl^2\sum_{k=0}^{l}\binom{l}{k}\Big(\prod_{i=l-k}^{l}4\beta^2e^2t_i^{-2}\Big)(4\log(n))^{4(l-k)}\ .$$

Now fix $0<\delta<\frac{1}{2}$. For l large, e.g. close to $L(n)$, the dominating factor in the sum above is the factor

$$\prod_{i=0}^{l}t_i^{-2}=o(t_L^{-r}),\ \text{for any } r>2.$$

Fix such an r then by (1) and (6) we have that $\prod_{i=0}^{l}t_i^{-2}\le c_1n^{\beta c_2}=o(n^\delta)$, for suitable constants c_1,c_2 and β small enough. If l is small the sum is dominated by the factor $\log(n)^{4l}$, which is also $o(n^\delta)$. Thus we get that the sum above is $o(n^{1+\delta})$ in any case. This also yields that

$$\mathbb{P}(G^c)\sum_{i=t_l n}^{n}c_i\le Cn^{3+\delta}e^{-\log(n)^2}\to 0\ .$$

By the triangle inequality,

$$\begin{aligned}&\mathbb{P}\big(|X_l-\mathbb{E}\,[X_l]\,|\ge n^{\frac{1}{2}+\delta}\big)\\ \le&\mathbb{P}\Big(|X_l-\mathbb{E}[f(Y_{\hat n_l+1}^{\mathrm{in}},\dots,Y_n^{\mathrm{out}})\,|\,(Y_{\hat n_l+1}^{\mathrm{in}},\dots,Y_n^{\mathrm{out}})\in G]|\ge n^{\frac{1}{2}+\delta}/2\Big)\\ &+\mathbb{P}\Big(|\mathbb{E}\,[X_l]-\mathbb{E}[f(Y_{\hat n_l+1}^{\mathrm{in}},\dots,Y_n^{\mathrm{out}})\,|\,(Y_{\hat n_l+1}^{\mathrm{in}},\dots,Y_n^{\mathrm{out}})\in G]|\ge n^{\frac{1}{2}+\delta}/2\Big)\ .\end{aligned}\tag{10}$$

By McDiarmid's inequality (8) the first term above is bounded from above by

$$2\mathbb{P}(G^c)+2\exp\left(-\frac{n^{1+2\delta}+o(1)}{2o(n^{1+\delta})}\right)\overset{n\to\infty}{\longrightarrow}0\ .$$

For the second term in (10) note that since we consider the conditional expectation of an event, the event inside the probability is not random. Write $Y=(Y_{\hat n_l+1}^{\mathrm{in}},\dots,Y_n^{\mathrm{out}})$ for short and note that by (9)

$$\begin{aligned}\mathbb{E}\,[X_l]-\mathbb{E}\,[f(Y)|Y\in G]=&\mathbb{E}\,[f(Y)\mathbb{1}_{Y\in G}]+\mathbb{E}\,[f(Y)\mathbb{1}_{Y\notin G}]-\mathbb{E}\,[f(Y)|Y\in G]\\ \le&\mathbb{E}\,[f(Y)|Y\in G]+\mathbb{P}(G^c)\sup_y f(y)-\mathbb{E}\,[f(Y)|Y\in G]\\ \le&Cn^{2+l}e^{-\log(n)^2}\overset{n\to\infty}{\longrightarrow}0\ ,\end{aligned}$$

since $l=O(\log\log n)$ and therefore $X_l=\mathbb{E}\,[X_l]+o(n/\log\log n)$ with high probability. Summing over l the upper bound in Theorem 1 follows.

For the lower bound we make use of Theorem 10 in [2]. To this end we let $S\subset(0,1]$ be an interval and introduce, for a continuous, symmetric kernel $\kappa\colon S^2\to[0,\infty)$, the inhomogeneous random graph $\mathcal{G}_n^\kappa$ obtained by connecting any pair of distinct vertices i,j such that $i/n,j/n\in S$ with probability $\frac{1}{n}\kappa(\frac{i}{n},\frac{j}{n})\wedge 1$. Any vertices i with $i/n\notin S$ are not present in the graph $\mathcal{G}_n^\kappa$. Moreover let T_κ be the operator on the space of bounded measurable functions $f\colon S\to\mathbb{R}$, given by

$$T_\kappa(f)(x)=\int_S\kappa(x,y)f(y)\mathrm{d}y\ .$$

Now we restate said theorem for the readers' convenience.

Lemma 2. *Let* $0 \leq t < 1$ *and* $\kappa\colon [t,1]^2 \to (0,\infty)$ *a symmetric continuous function. Let* $c > 0$ *be a constant and* $\phi \in C[0,1]$ *strictly positive such that* $c\,T_\kappa(\phi)(x) \geq \phi(x)$*, for all* $x \in [t,1]$*. Let* $0 < \varepsilon < 1$ *and* $c' = (1+\varepsilon)c \leq 1$ *and define* $\mathcal{G}_n^\kappa(c')$ *as the graph, which is obtained from* $\mathcal{G}_n^\kappa$ *by keeping every edge independently with probability* c'*. Then, with high probability,* $\mathcal{G}_n^\kappa(c')$ *contains a component of order at least* $Cn - o(n)$*, where*

$$C = \Big(\frac{\varepsilon}{1+\varepsilon}\Big)\,\frac{\int_t^1 \phi(x)\,dx}{\sup_{t\leq x\leq 1}\phi(x)}\ .$$

Note that in the original source the authors consider only the interval $[0,1]$, but going through the proof of the theorem reveals that this also holds in the more general form above. Let $\kappa\colon [t,1]^2 \to (0,\infty)$ be given by $\kappa(x,y) = \beta(x\wedge y)^{-\gamma}(x\vee y)^{\gamma-1}$. Therefore we now need to find a constant $c \in (0,1)$ and a function ϕ such that $c\,T_\kappa(\phi)(x) \geq \phi(x)$ for all $x \in [t,1]$. We make an educated guess and choose $\phi(x) = x^{\gamma-1}$, then we have

$$\begin{aligned} T_\kappa(\phi)(x) =& \beta x^{\gamma-1}\int_t^x y^{-\gamma}y^{\gamma-1}\mathrm{d}y + \beta x^{-\gamma}\int_x^1 y^{\gamma-1}y^{\gamma-1}\mathrm{d}y \\ =& \beta x^{\gamma-1}\left[\log(y)\right]_t^x + \beta x^{-\gamma}\frac{1}{2\gamma-1}\left[y^{2\gamma-1}\right]_x^1 = \beta x^{\gamma-1}\left[\log(x) - \log(t) + \frac{x^{1-2\gamma}}{2\gamma-1} - \frac{1}{2\gamma-1}\right]\ . \end{aligned}$$

By differentiating we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log(x) - \log(t) + \frac{x^{1-2\gamma}}{2\gamma-1} - \frac{1}{2\gamma-1}\right] = x^{-1} - x^{-2\gamma} \leq 0\ ,$$

since $\gamma > \frac{1}{2}$. Thus the term in brackets is monotone decreasing in x and therefore takes its minimal value at $x = 1$. Hence $T(\phi)(x) \geq \beta\log(\frac{1}{t})x^{\gamma-1}$. Now we need that $c < 1$, which is equivalent to $\beta\log(\frac{1}{t}) > 1$. That is why we choose our cutoff to be

$$t := \exp\Big(-\frac{1+\varepsilon}{\beta}\Big)\ ,$$

for some $\varepsilon > 0$ arbitrary. Moreover we have

$$\int_t^1 x^{\gamma-1}\mathrm{d}x = \frac{1}{\gamma}(1-t^\gamma) \overset{t\to 0}{\to} \frac{1}{\gamma} \quad \text{and} \quad \sup_{t\leq x\leq 1} x^{\gamma-1} = t^{\gamma-1}\ .$$

We apply Lemma 2 with $c = \frac{1}{1+\varepsilon}$, $c' = 1$ and get that we have, with high probability, a giant component of size at least

$$C_1(\mathcal{G}_n) \geq \left(\frac{1}{\gamma} + o(1)\right)\frac{\varepsilon}{1+\varepsilon}\,\left(t^{\gamma-1}\right)^{-1}\, n \geq \exp\left(-(1-\gamma)\frac{1+2\varepsilon}{\beta}\right)\, n\ ,$$

with the last inequality holding for β small enough. Since ε was arbitrary this completes the proof.

3 Proof of Proposition 1

We construct a path from an initial vertex to the core by connecting successively to more powerful vertices (i.e. vertices with lower index). Recall that $m := m(n) := \sqrt{n}\,(\log n)^{-\alpha}$, for $\alpha = (4-2\gamma)^{-1}$. The core consists of vertices with index $\leq m(n)$, which satisfy a condition on their degree. Inside the core distances are very small, in particular the diameter of the core is bounded by a constant independent of n. For the construction we use the graph consisting of $2n$ vertices, as two powerful vertices in $\mathcal{G}_{2n}$ are typically connected by a vertex in $\mathcal{G}_{2n}\setminus\mathcal{G}_n$. We start by defining the core and proving that it has bounded diameter.

Definition 1.

1. Let $x \neq y \in \{1, \dots, n\}$ be two vertices in $\mathcal{G}_n$. We call $u \in \{n+1, \dots, 2n\}$ a *1-connector* for x and y, if $x \leftrightarrow u$ and $u \leftrightarrow y$ in $\mathcal{G}_{2n}$. If x and y are connected by a 1-connector we write $x \overset{1}{\longleftrightarrow} y$.
2. We define the core as the graph $\mathsf{core}_n := (C_n, E_n)$ with

$$C_n := \left\{x \in \{1, \dots, m\} \,:\, \deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x) \geq \sigma \mathbb{E}[\deg_{\mathcal{G}_n}(x)]\right\} ,$$

where $\sigma = (1-\gamma)\gamma 2^{\gamma-2}$. The edges are given by

$$E_n := \left\{(x, y) \in C_n \times C_n \,:\, x \overset{1}{\longleftrightarrow} y\right\} .$$

Note that this is not a subgraph of $\mathcal{G}_{2n}$, however two connected vertices in core_n are also connected in $\mathcal{G}_{2n}$ by a two step connection. Before we prove that core_n has bounded diameter, let us first establish an auxiliary result, which we will use multiple times throughout this section.

Lemma 3.

1. *For $x \in \{1, \dots, n\}$ the probability that x satisfies the degree condition in C_n is uniformly bounded away from 0, i.e. there exists a constant $q > 0$ not depending on n or x such that*

$$\mathbb{P}\left(\deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x) \geq \sigma \mathbb{E}[\deg_{\mathcal{G}_n}(x)]\right) \geq q$$

2. *There exists a constant $\rho \in (0, 1)$, not depending on n, such that with high probability*

$$m \geq \#C_n \geq \rho m.$$

Proof. We first show that there exists a $\theta \in (0, 1)$ such that

$$\sigma \mathbb{E}[\deg_{\mathcal{G}_n}(x)] \leq \theta \mathbb{E}[\deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x)] \,. \tag{11}$$

We have that

$$\sigma \mathbb{E}[\deg_{\mathcal{G}_n}(x)] \leq \sigma \beta \left(\frac{x^{-\gamma} n^{\gamma}}{\gamma} + \frac{1}{1-\gamma}\right)$$

and also

$$\mathbb{E}[\deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x)] \geq \beta 2^{\gamma-1} x^{-\gamma} n^{\gamma} \,.$$

Hence it sufficient to choose

$$\theta = \sigma 2^{1-\gamma}\left(\frac{1}{\gamma} + \frac{1}{1-\gamma}\right),$$

which is strictly smaller than 1 by choice of σ. Furthermore let $Z := Z(x) := \deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x) \geq 0$. It is clear that Z is dominated by a binomial random variable with parameters n and $\tilde{p} = \beta x^{-\gamma} n^{\gamma-1}$ and thus $\mathbb{E}[Z^2] \leq n^2 \tilde{p}^2 + n\tilde{p}$. Therefore by (11), the Paley-Zygmund inequality and our moment estimates for Z, in that order, we have

$$\begin{aligned}
&\mathbb{P}\big(\deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x) \geq \sigma \mathbb{E}[\deg_{\mathcal{G}_n}(x)]\big) \\
&\qquad \geq \mathbb{P}\left(Z \geq \theta \mathbb{E}[Z]\right) \geq (1-\theta)^2 \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} \\
&\qquad \geq (1-\theta)^2 \frac{(2^{\gamma-1} n\tilde{p})^2}{n^2\tilde{p}^2 + n\tilde{p}} \geq (1-\theta)^2 \frac{4^{\gamma-1}}{1+\beta^{-1}} =: q,
\end{aligned}$$

which proves the first statement. For the second, note that the first inequality is trivial. By the first statement $\#C_n$ dominates a binomial random variable with parameters m and q. Hence, by Hoeffding's inequality, for all $t > 0$,

$$\mathbb{P}\left(\#C_n \leq qm - t\right) \leq \mathbb{P}\left(|\mathrm{Bin}(m, q) - qm| \geq t\right) \leq 2\exp\left(-\tfrac{2t^2}{m}\right) \,.$$

For $t = qm/2$ the term above goes to 0 as $m =\!\!\rightarrow \infty$, which implies the second statement for $\rho = q/2$.

Lemma 4. *For $m = m(n) = \sqrt{n}\log(n)^{-\alpha}$, with $\alpha = \frac{1}{4-2\gamma}$, the graph core_n as defined above has, with high probability, bounded diameter independent of n.*

Proof. We prove the statement by comparing core_n with a suitable Erdős-Rényi graph. For $x, y \in C_n$ we want to lower bound the connection probability $\mathbb{P}((x,y) \in E_n)$. Thus let $\mathcal{Z} := \mathcal{Z}(x)$ be the set of all potential 1-connectors for x and y with an edge to x: $\mathcal{Z}(x) := \{u \in \{n+1, \dots, 2n\} \,:\, u \leftrightarrow x\}$. By definition of the core we have that

$$\#\mathcal{Z} \geq \sigma\mathbb{E}[\deg_{\mathcal{G}_n}(x)] \geq \sigma n \beta m^{-\gamma} n^{\gamma-1} = \sigma\beta\left(\frac{n}{m}\right)^{\gamma} .$$

Hence

$$\begin{aligned}\mathbb{P}((x,y) \notin E_n) = \prod_{v \in \mathcal{Z}(x)} (1 - \beta y^{-\gamma} v^{\gamma-1}) &\leq (1 - \beta m^{-\gamma}(2n)^{\gamma-1})^{\sigma\beta(\frac{n}{m})^{\gamma}} \\ &\leq \exp\left(-\sigma\beta^2 m^{-2\gamma} 2^{\gamma-1} n^{2\gamma-1}\right) \leq 1 - \frac{\sigma}{2}\beta^2 m^{-2\gamma} 2^{\gamma-1} n^{2\gamma-1},\end{aligned}$$

where we used that $e^{-z} \leq 1 - z/2$ for z small enough. Therefore

$$\mathbb{P}((x,y) \in E_n) \geq \frac{\sigma}{2}\beta^2 m^{-2\gamma} 2^{\gamma-1} n^{2\gamma-1} =: p .$$

Now we use the Erdős-Rényi graph $\mathcal{G}(N,p)$, with $N := \#C_n$ and get $\operatorname{diam}(\mathcal{G}(N,p)) \geq \operatorname{diam}(\mathsf{core}_n)$ with high probability, by construction. To see that $\mathcal{G}(N,p)$ has bounded diameter we use the following result from [1], see Corollary 10 there.

Lemma 5. *Suppose $d = d(N) > 2$ and $0 < p = p(N) < 1$ satisfy $(\log N)/d - 3\log\log N \to \infty$, $p^d N^{d-1} - 2\log N \to \infty$ and*

$$(\log N)(p^{d-1}N^{d-2} - \log N + \log\log N) \to -\infty .$$

Then $\mathcal{G}(N,p)$ has, with high probability, diameter $d(N)$.

By Lemma 3 we have that $m \geq \#C_n \geq \rho m$ and thus

$$m = m(n) = \frac{\sqrt{n}}{(\log n)^{\frac{1}{4-2\gamma}}} \quad \text{and} \quad d = \frac{2}{2\gamma - 1}$$

satisfy the assumptions of the above lemma, which completes the proof.

As mentioned earlier we are aiming to construct a path from any vertex in the giant component into the core. To this end we introduce *layers* of higher and higher connected vertices using a sequence $(t_k)_{k\in\mathbb{N}_0}$ similar to the last section. Here we take inspiration from estimate (6), letting

$$t_k := \delta \, \exp\left(-\frac{1}{\beta^2}\left(\frac{\gamma}{1-\gamma}\right)^k\right) , \quad \text{for some } \delta \in (0,1) .$$

We want to identify good vertices in every layer and construct a path connecting a good vertex in layer k to a good vertex in layer $k+1$.

Definition 2. 1. For all $0 \leq k < n$ we define the *$k-th$ layer* in $\mathcal{G}_{2n}$ as

$$\mathcal{N}^{(k)} := \{\lfloor t_{k+1} n \rfloor, \dots, \lceil t_k n \rceil\} .$$

2. We call a vertex *good* if $x \in \mathcal{N}^{(k)}$ for some $0 \leq k < n$ and

$$\deg_{\mathcal{G}_{2n}}(x) - \deg_{\mathcal{G}_n}(x) \geq \sigma\mathbb{E}[\deg_{\mathcal{G}_n}(x)] .$$

We first need to find a good vertex around our initial vertex. In [6] (see also Section 2 of [12]) a coupling of the rooted graph $(\mathcal{G}_{2n}, U)$ to a branching random walk $\mathfrak{T}$ is constructed, where U is a distinguished vertex called the *root* chosen uniformly at random from $\mathcal{G}_{2n}$ and $\mathfrak{T}$ is a branching random walk on the negative half-axis with a killing barrier at 0, initial particle at $-X$, where X is standard exponential, and Poisson offspring with intensity measure π given by

$$\pi(\mathrm{d}y) = \beta\big(e^{(1-\gamma)y}\mathbb{1}_{y<0} + e^{\gamma y}\mathbb{1}_{y>0}\big)\mathrm{d}y\ .$$

The coupling is such that, with high probability, a local neighbourhood exploration of U in $\mathcal{G}_{2n}$ and of $-X$ in $\mathfrak{T}$ up to a finite number of steps yield the same graph. Additionally, particles of $\mathfrak{T}$ at position $x<0$ are mapped to vertex $i \in \mathcal{G}_{2n}$ iff

$$-\sum_{j=i}^{2n}\frac{1}{j} < x \leq -\sum_{j=i+1}^{2n}\frac{1}{j}.$$

Lemma 6. *Let $U \in \mathcal{G}_{2n}$ be a uniformly chosen vertex and let $\mathcal{C}_{2n}^K(U)$ denote the local neighbourhood exploration around U after K steps. Then, for every $\varepsilon > 0$ there exists a $K = K(\varepsilon) \in \mathbb{N}$ such that*

$$\mathbb{P}\left(\mathcal{C}_{2n}^K(U) \text{ contains a good vertex}\right) \geq \mathbb{P}(|\mathfrak{T}| = \infty) - \varepsilon\ ,$$

where $\mathbb{P}(|\mathfrak{T}| = \infty)$ is the survival probability of the killed branching random walk.

Proof. We introduce the notation

$$E_k = \{\text{the coupling fails before step } k+1\} \quad \text{for } k \in \mathbb{N}\ .$$

A good vertex in the graph is coupled to a particle in $\mathfrak{T}$, with a position to the left of $\log(t_0)$ and a certain (large) number of offspring in $[\log(\frac{1}{2}), 0)$. Therefore we can equivalently search for *good particles* in $\mathfrak{T}$. For $k \in \mathbb{N}$ we define $\mathfrak{T}^k$ as the killed branching random walk, which we get after the first k explored vertices in $\mathfrak{T}$. Then

$$\begin{aligned} &\mathbb{P}\left(\mathcal{C}_{2n}^K(U) \text{ contains a good vertex}\right) \\ \geq &\mathbb{P}\left(\mathfrak{T}^K \text{ contains a good particle}\right) - \mathbb{P}\left(E_K\right) \\ \geq &\mathbb{P}(|\mathfrak{T}| = \infty)\mathbb{P}\left(\mathfrak{T}^K \text{ contains a good particle}\,|\,|\mathfrak{T}| = \infty\right) - \mathbb{P}\left(E_K\right)\ . \end{aligned} \tag{12}$$

We need the following fact for our argument

$$|\mathfrak{T}| = \infty \quad \Longleftrightarrow \quad \forall L < \infty \text{ we have } |\mathfrak{T} \cap (-\infty, -L]| = \infty \quad \text{almost surely.} \tag{13}$$

The implication $\Leftarrow$ is trivial. The other we prove by contradiction. By survival there are particles in every generation. Assume we have one generation G after which no more offspring in $(-\infty, -L]$ occur, for some $L < \infty$. Let $(y_1, y_2, \dots)$ denote particles, not necessarily descendants of one another, such that y_i is in generation $G+i$. Then every y_i has positive probability $p = 1 - \exp(-\pi((-\infty, -L)))$ to produce offspring to the left of $-L$. Hence we have

$$\mathbb{P}\left(\forall i \in \mathbb{N} : y_i \text{ has no offspring in } (-\infty, -L)\right) = \prod_{i=1}^{\infty}(1-p) = 0\ .$$

This is a contradiction, which proves statement (13).

Let $b = \log(t_0)$ and note that for every $M \in \mathbb{N}$ we have for the conditional probability in (12)

$$\begin{aligned} \mathbb{P}\big(\mathfrak{T}^K \text{ contains a good particle}\,|\,|\mathfrak{T}| = \infty\big) \geq \mathbb{P}\left(|\mathfrak{T}^K \cap (-\infty, b)| \geq M\,|\,|\mathfrak{T}| = \infty\right) \cdot \\ \mathbb{P}\left(\mathfrak{T}^K \text{ contains a good particle}\,|\,|\mathfrak{T}^K \cap (-\infty, b)| \geq M\,,\, |\mathfrak{T}| = \infty\right)\ . \end{aligned}$$

By Lemma 3 every particle to the left of $b = \log(t_0)$ has a uniform positive probability to be good. So we find M not depending on K such that

$$\mathbb{P}\left(\mathfrak{T}^K \text{ contains a good particle} \,\middle|\, |\mathfrak{T}^K \cap (-\infty, b)| \geq M\,,\ |\mathfrak{T}| = \infty\right) \geq 1 - \tfrac{\delta}{2}.$$

Moreover, by (13) we can pick $K = K(M)$ such that

$$\mathbb{P}\left(|\mathfrak{T}^K \cap (-\infty, b)| \geq M \,\middle|\, |\mathfrak{T}| = \infty\right) \geq 1 - \tfrac{\delta}{2}\ .$$

Then we get by the above that

$$\mathbb{P}\left(\mathfrak{T}^K \text{ contains a good particle} \,\middle|\, |\mathfrak{T}| = \infty\right) \geq (1 - \tfrac{\delta}{2})^2 \geq 1 - \delta\ .$$

Now we choose $\delta = \frac{\varepsilon}{2\mathbb{P}(|\mathfrak{T}|=\infty)}$ and thus get from (12) that

$$\begin{aligned}\mathbb{P}\left(\mathcal{C}_{2n}^K(x) \text{ contains a good vertex}\right) &\geq \mathbb{P}(|\mathfrak{T}| = \infty)(1-\delta) - \mathbb{P}(E_K)\\ &\geq \mathbb{P}(|\mathfrak{T}| = \infty) - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}\ .\end{aligned}$$

As mentioned earlier we now construct a path along good vertices into the core. We let $\mathcal{F}_0$ be the σ-algebra generated by the exploration of a uniformly chosen vertex $U \in \mathcal{G}_n$ stopped either at the *good event* when the first good vertex V is found or at the *bad event* when the coupling fails or we have explored K vertices without finding a good vertex.

Lemma 7. *Let $\varepsilon > 0$. There exists a constant $C \leq 0$ such that on the good event*

$$\mathbb{P}\Big(d_{2n}(V, \mathsf{core}_n) > 2\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + 2C \,\Big|\, \mathcal{F}_0\Big) \leq \varepsilon\,,$$

if n is sufficiently large.

Proof. We iteratively construct a path from V to the core. Assuming we have found a good vertex $V_k \in \mathcal{N}^{(k)}$ we aim to connect it to a good vertex in $V_{k+1} \in \mathcal{N}^{(k+1)}$ using a 1-connector. The good event ensures that we can start this construction with $V \in \mathcal{N}^{(k)}$, without loss of generality we can assume $k = 0$. Note that in every step this adds two edges to our path. Let $\mathcal{F}_k$ be the σ-algebra generated by our construction up until we found V_k, for $k = 0$ this coincides with $\mathcal{F}_0$ from the conditions of the lemma. Let $\mathcal{Z}(V_k) := \{u \in \{n+1, \ldots, 2n\} \,:\, u \leftrightarrow V_k\}$, then we have

$$\begin{aligned}&\mathbb{P}(\nexists \text{ good } w \in \mathcal{N}^{(k+1)} \,:\, w \overset{1}{\leftrightarrow} V_k \,|\, \mathcal{F}_k)\\ &=\mathbb{E}\Big[\prod_{w\in\mathcal{N}^{(k+1)}} \mathbb{P}\big(\{w \text{ not good}\} \cup \{\nexists i \in \mathcal{Z}(V_k) \,:\, i \leftrightarrow w \text{ and } w \text{ good}\}\big) \,\Big|\, \mathcal{F}_k\Big]\\ &\leq\mathbb{E}\Big[\prod_{w\in\mathcal{N}^{(k+1)}} \Big(\mathbb{P}\big(w \text{ not good}\big) + \mathbb{P}\big(w \text{ good}\big) \prod_{i\in\mathcal{Z}(V_k)} \mathbb{P}\big(i \not\leftrightarrow w \,|\, \{w \text{ good}\}, \bigcap_{\substack{j\in\mathcal{Z}(V_k):\\ j<i}} \{j \not\leftrightarrow w\}\big)\Big) \,\Big|\, \mathcal{F}_k\Big].\end{aligned}$$

By Lemma 3, $\mathbb{P}(w \text{ good}) \geq q$ and

$$\mathbb{P}\big(i \not\leftrightarrow w \,|\, \{w \text{ good}\}, \bigcap_{\substack{j\in\mathcal{Z}(V_k):\\ j<i}} \{j \not\leftrightarrow w\}\big) \leq 1 - \frac{\beta 2^{\gamma-1} t_{k+1}^{-\gamma}}{n}.$$

Plugging this in, we get

$$\begin{aligned}&\mathbb{P}(\nexists \text{ good } w \in \mathcal{N}^{(k+1)} \,:\, w \overset{1}{\leftrightarrow} V_k \,|\, \mathcal{F}_k) \leq \Big(1 - q + q\,\big(1 - \tfrac{\beta 2^{\gamma-1} t_{k+1}^{-\gamma}}{n}\big)^{\sigma\mathbb{E}[\deg_{\mathcal{G}_n}(v)]}\Big)^{|\mathcal{N}^{(k+1)}|}\\ &\leq \exp\Big(-c\frac{\beta^2 t_{k+1}^{-\gamma}}{n} t_k^{-\gamma}(t_k - t_{k+1})n\Big) \leq \exp\Big(-c\beta^2(t_{k+1}^{-\gamma} t_k^{1-\gamma} - t_{k+1}^{1-\gamma} t_k^{-\gamma})\Big)\\ &\leq \exp\Big(-c\,\delta^{1-2\gamma}\,\tau\,\big(\tfrac{\gamma}{1-\gamma}\big)^k\Big),\end{aligned}$$

for a suitable constant c, where the last step follows by series expansion of the exponential function. Furthermore the constant in the above is given by

$$\tau = \tau(\gamma) = (1-\gamma)\left(\left(\tfrac{\gamma}{1-\gamma}\right)^2 - 1\right) > 0 \,.$$

Thus we get by summing over all k

$$\mathbb{P}(\text{the construction fails}) \leq \sum_{k=0}^{\infty} \exp\left(-c\,\delta^{1-2\gamma}\,\tau\,\left(\tfrac{\gamma}{1-\gamma}\right)^k\right) =: \sum_{k=0}^{\infty} a_k \,.$$

It is clear that the sequence $(\frac{a_{k+1}}{a_k})_{k\in\mathbb{N}_0}$ is strictly monotonically decreasing, with $\frac{a_{k+1}}{a_k} \to 0$ as $k \to \infty$. Hence for $r = \frac{a_1}{a_0}$ we have that $a_{k+1} \leq r a_k$ for all $k \geq 0$. Therefore we get

$$\sum_{k=0}^{\infty} a_k = a_0 + \sum_{k=1}^{\infty} a_k \leq a_0 + \sum_{k=1}^{\infty} r^k a_0 = a_0 + a_0\,\frac{r}{1-r} \,.$$

Since δ was arbitrary, we can choose

$$\delta = \delta(\varepsilon) = \left(\frac{-\log(\frac{\varepsilon}{2})}{c\,\tau}\right)^{\frac{1}{1-2\gamma}} \in (0,1) \quad \text{for } \varepsilon \text{ small enough.}$$

Thus by the above

$$\mathbb{P}(\text{the construction fails}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\,\frac{r}{1-r} \leq \varepsilon \,,$$

for $\varepsilon > 0$ arbitrary. Which means we can carry on our construction until we reach the core, that is until $t_L n \leq \sqrt{n}\log(n)^{-\alpha}$ by Lemma 4 and $\alpha = \frac{1}{4-2\gamma}$. This is satisfied by

$$L = L(n) = \frac{\log\log(\sqrt{n}\log(n)^{\alpha})}{\log\left(\frac{\gamma}{1-\gamma}\right)} + C \,, \quad \text{where } C = \frac{\log(\beta^2)}{\log\left(\frac{\gamma}{1-\gamma}\right)} \,. \tag{14}$$

Recalling the definition of the core, see Definition 1, we see that a good vertex in layer L is also in the core for n large enough. As $\log\log(\sqrt{n}\log(n)^{\alpha}) = \log\log(n) + \log\frac{1}{2} + o(1)$ this completes the proof.

Thus we have acquired all the tools necessary to prove Proposition 1. Fix $\varepsilon > 0$. Now let U, V be independent, uniform random variables on $\{1,\dots,n\}$. Then,

$$\begin{aligned}
&\frac{1}{n^2}\#\Big\{(x,y) \in \mathcal{C}_1(\mathcal{G}_n)^2 \colon d_n(x,y) \geq 4\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + B\Big\} \\
=&\mathbb{P}\big(U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\big)\mathbb{P}\Big(d_n(U,V) \geq 4\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + B \,\Big|\, U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\Big)
\end{aligned}$$

The first probability is strictly greater than 0 since $\gamma > 1/2$ and can be bounded from above by 1. Let $\mathcal{F}_0$ be the σ-algebra generated by the exploration around U and V until we are stopped by either the good or bad event. On the good events we found good vertices u and v in at most $K = K(\varepsilon)$ steps. Recalling $L(n)$ from (14) we have

$$\begin{aligned}
&\mathbb{P}\big(d_n(U,V) < 4L(n) + B \,\big|\, U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\big) \\
&\qquad = \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{P}\big(d_n(U,V) < 4L(n) + B \,\big|\, U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\big)\Big|\mathcal{F}_0\Big]\Big] \\
&\qquad \geq \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{P}\left(\mathcal{C}_{2n}^K(U), \mathcal{C}_{2n}^K(V) \text{ contain good } u,v \,\middle|\, U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\right) \cdot \\
&\qquad\qquad \mathbb{P}\Big(d_n(u,v) < 4L(n) + B \,\Big|\, \mathcal{C}_{2n}^K(U), \mathcal{C}_{2n}^K(V) \text{ contain good } u,v,\ U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\Big)\Big|\mathcal{F}_0\Big]\Big]
\end{aligned}$$

By Lemma 6 the first term is greater or equal $1-\varepsilon$. In the second term since u, v are good they can each be connected to the core in $2L$ steps with probability greater then $1-2\varepsilon$, if n is sufficiently large by Lemma 7. Finally the diameter of the core is bounded by some constant d by Lemma 4 with high probability. Hence $\mathbb{P}(\text{diam}(\text{core}_n) \leq d) \geq 1-\varepsilon$, for n large enough. All in all we have

$$\mathbb{P}\Big(d_n(U,V) \geq 4\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + 2K + 2C + d \,\Big|\, U,V \in \mathcal{C}_1(\mathcal{G}_n)^2\Big) \leq 4\varepsilon \ ,$$

for all $\varepsilon > 0$, if n is sufficiently large. Which gives us

$$\frac{1}{n^2}\#\Big\{(x,y) \in \mathcal{C}_1(\mathcal{G}_n)^2 \colon d_n(x,y) \geq 4\frac{\log\log(n)}{\log\left(\frac{\gamma}{1-\gamma}\right)} + B\Big\} \to 0 \ .$$

Remark. Together with Theorem 2 in [7] we get that the upper bound in Proposition 1 is sharp.

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