The parabolic Anderson model with heavy-tailed potential

Peter Mörters

joint work with

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Hubert Lacoin (Paris)
Marcel Ortgiese (Bath)
Nadia Sidorova (London)

The project

Aim: Study diffusion in a random medium or potential.

Questions:

- Which qualitative effects can be caused by small inhomogeneities in the medium?
- Which qualitative effects can be caused by considerable irregularity of the medium?

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This talk will focus on the second question, but we will start with a general introduction of the parabolic Anderson model.

The parabolic Anderson problem

The parabolic Anderson problem is the Cauchy problem for the heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t,z) &= \Delta u(t,z) + \xi(z) u(t,z), && \text{for } (t,z) \in [0,\infty) \times \mathbb{Z}^d, \\ u(0,z) &= \mathbf{1}_0(z), && \text{for } z \in \mathbb{Z}^d, \end{aligned}$$

with

discrete Laplacian $(\Delta f)(z) = \sum_{y \sim z} [f(y) - f(z)]$ and

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random potential $\{\xi(z) \colon z \in \mathbb{Z}^d\}$ independent, identically distributed.

The problem has a unique nonnegative solution if

$$E[(\xi(0) \vee 0)^{d+\varepsilon}] < \infty$$

for some $\varepsilon > 0$, which will always be assumed.

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The (random) solution of the parabolic Anderson problem is given by the expected mass at time t at site z. This is the content of the celebrated Feynman–Kac formula

$$u(t,z) = \mathbb{E}_0 \Big\{ \mathbf{1}_{\{X_t = z\}} \exp\Big(\int_0^t \xi(X_s)\, ds \Big) \Big\} \qquad \text{for } t > 0, z \in \mathbb{Z}^d.$$

Intermittency effect

For any nondegenerate potential distribution, the parabolic Anderson model is believed to exhibit an intermittency effect:

As time progresses, the bulk of the mass of the solution is not spreading in a regular fashion, but becomes concentrated in a small number of spatially separated islands of moderate size determined by the potential.

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Heuristics: In the Feynman-Kac formula

$$\sum_{z\in\mathbb{Z}^d} u(t,z) = \mathbb{E}_0\Big\{ \exp\Big(\int_0^t \xi(X_s)\, ds\Big)\Big\}.$$

there is a competition between the benefits of spending much time at sites with large potential values and the unlikeliness of this behaviour. The paths $(X_s\colon 0 \le s \le t)$ that give the dominant contribution to the integral are likely to end in certain regions of the lattice, the islands.

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Main contributors in this research area: Molchanov, Gärtner, König, Sznitman, den Hollander, ... but there are still many open problems.

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In this talk we focus on a case of heavy tails and derive fine properties of the solution, including a detailed discussion of the number of islands in which the solution is concentrated.

Heavy tailed potentials

We now assume that $\xi(0)$ is Pareto-distributed, i.e.

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Questions:

- How many sites are needed to support the bulk of the solution?
- Where are these sites?
- How fast does the solution grow?

Complete localisation

Theorem 1 (König, Lacoin, M, Sidorova 2006)

There exists a stochastic process $(Z_t \colon t > 0)$ with values in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t, Z_t)}{\sum_{z\in\mathbb{Z}^d} u(t,z)} = 1 \quad \text{in probability.}$$

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$$\Big(\Big(\frac{\log t}{t}\Big)^{\frac{\alpha}{\alpha-d}} Z_{st}, \Big(\frac{\log t}{t}\Big)^{\frac{d}{\alpha-d}} \frac{\log U(st)}{st} \; : \; s>0 \Big)$$

$$\Rightarrow \big(Y_s^{(1)}, Y_s^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{s}\big)\|Y_s^{(1)}\| \; : \; s>0 \big),$$

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Extreme value theory approach

Recall that

$$\frac{1}{t}\log U(t) \approx \max_{z\in\mathbb{Z}^d} \Psi_t(z)$$

for

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For $r_t = (t/\log t)^{\frac{\alpha}{\alpha-d}}$ and $a_t = (t/\log t)^{\frac{d}{\alpha-d}}$ the point process

$$\Pi_t = \sum_{z\in\mathbb{Z}^d} \delta_{(\frac{z}{r_t},\frac{\Psi_t(x)}{a_t})}$$

converges to a Poisson process with intensity measure

$$\nu(\mathrm{d}x\,\mathrm{d}y) = \mathrm{d}x \otimes \frac{\alpha\,\mathrm{d}y}{(y + \frac{d}{\alpha-d}\|x\|)^{\alpha+1}}\,.$$

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For fixed s and large t we obtain

$$\frac{\Psi_{st}(z)}{a_t} \approx \frac{\Psi_t(z)}{a_t} + \frac{d}{\alpha-d}\left(1-\tfrac{1}{s}\right)\frac{\|z\|}{r_t}\,.$$

Definition of the limit process

Let Π be a Poisson point process with intensity measure

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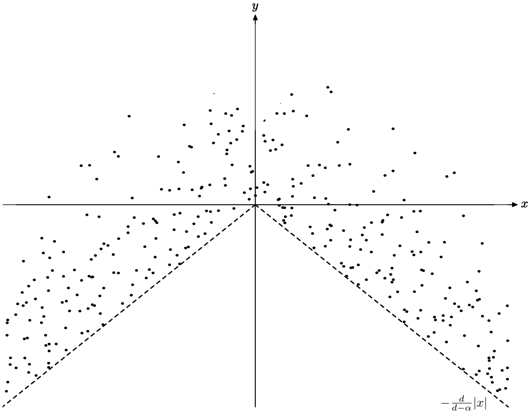

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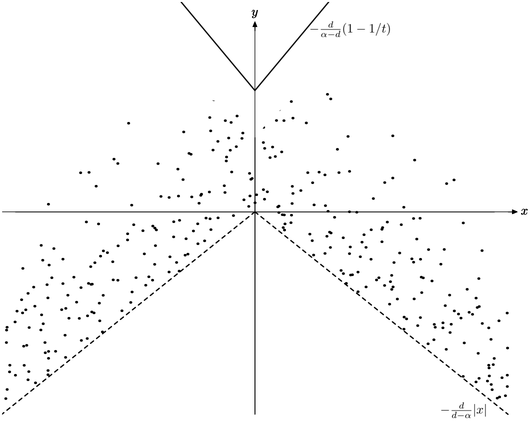

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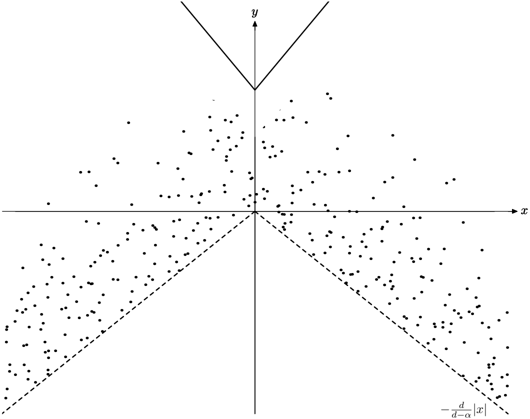

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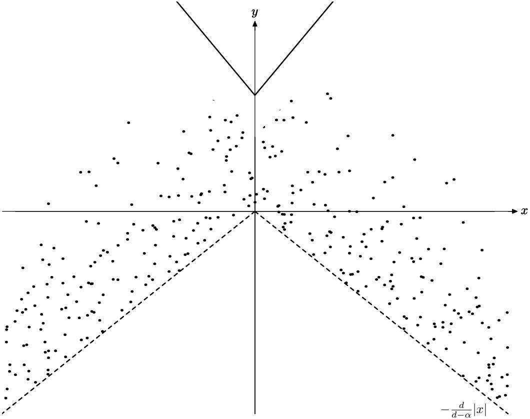

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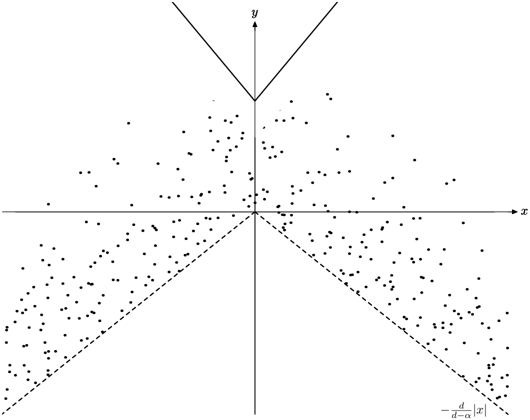

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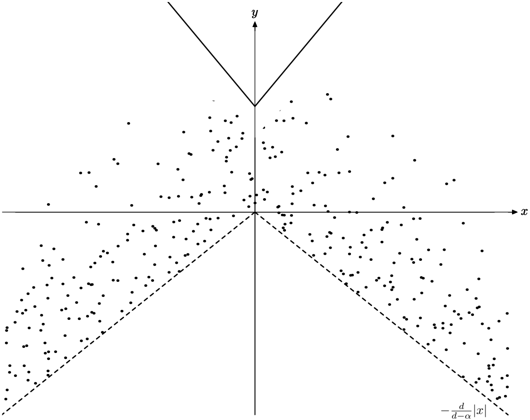

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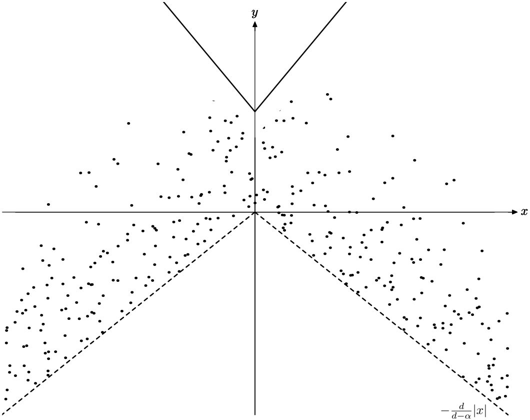

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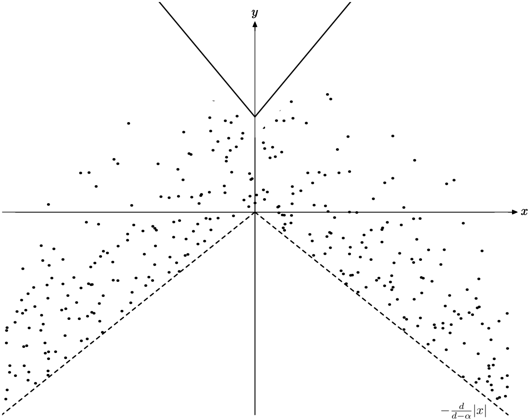

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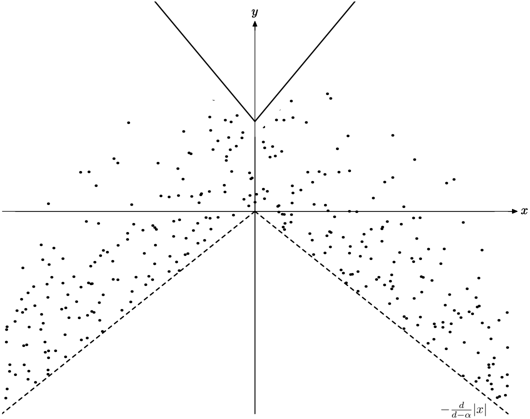

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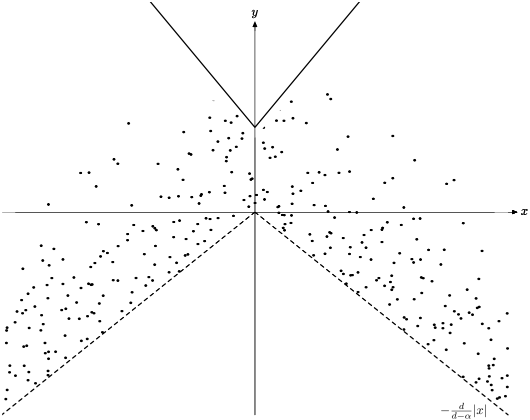

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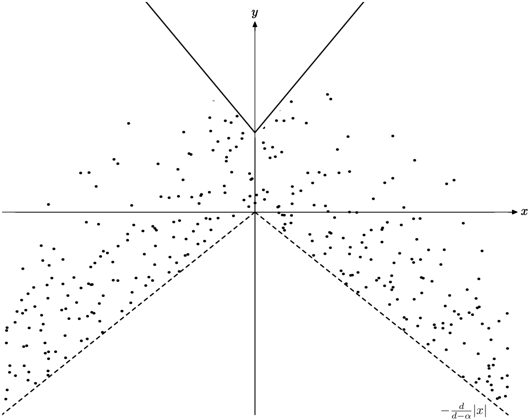

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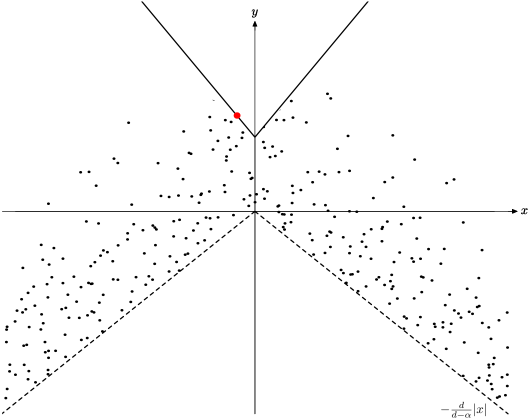

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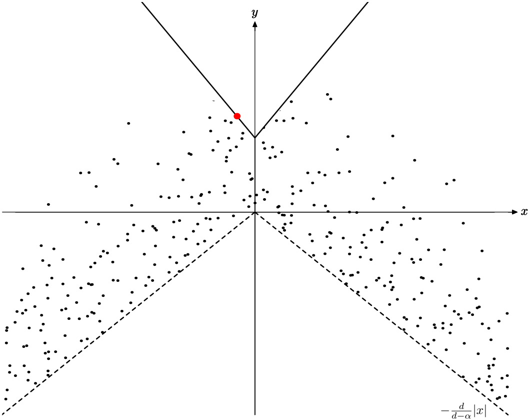

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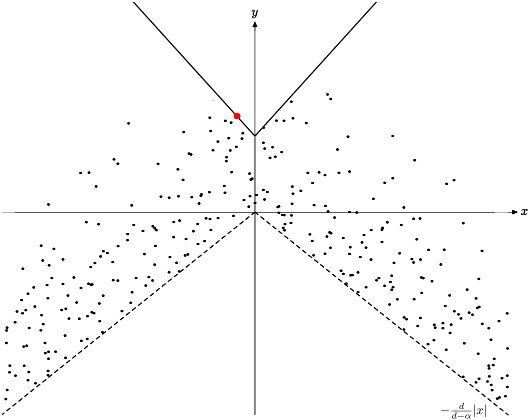

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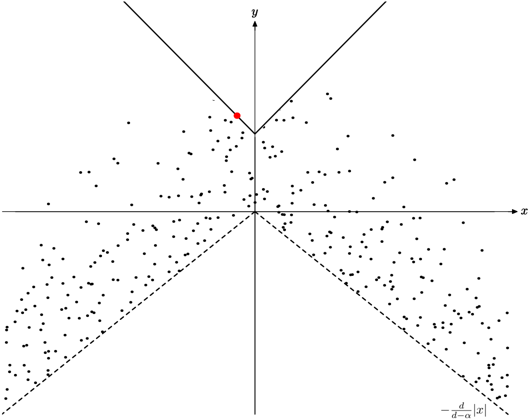

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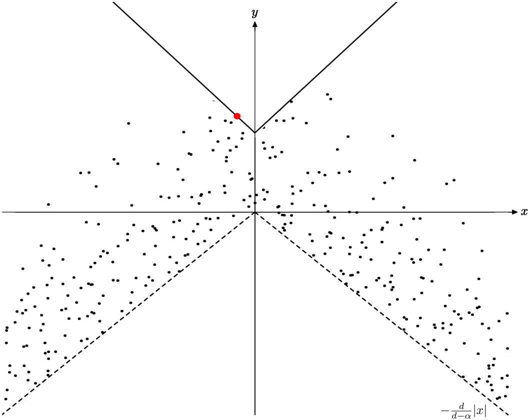

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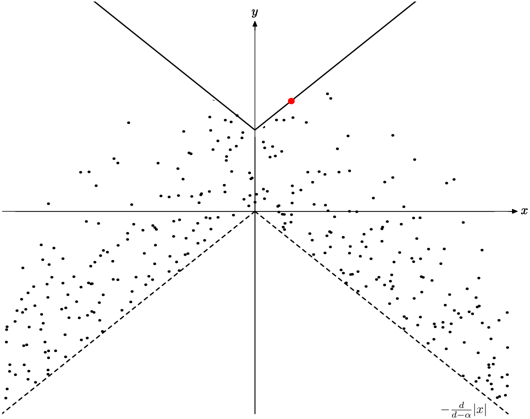

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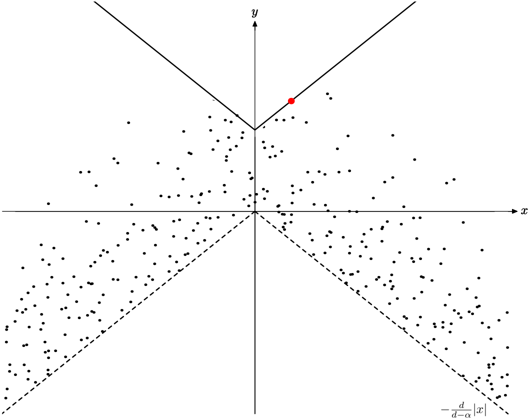

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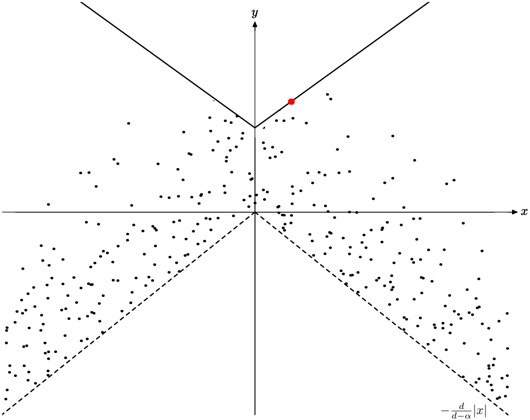

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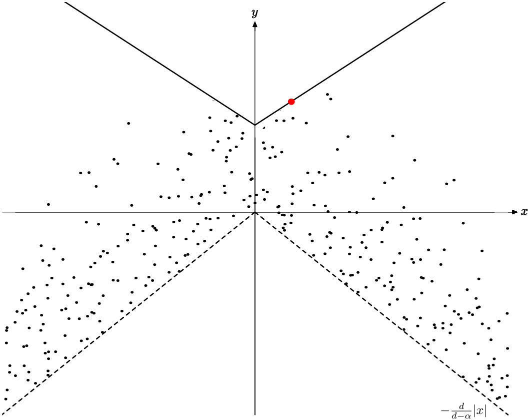

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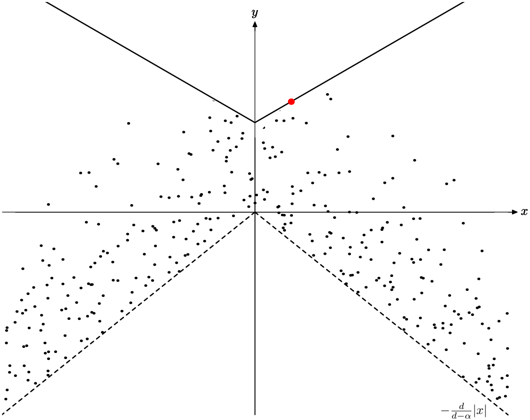

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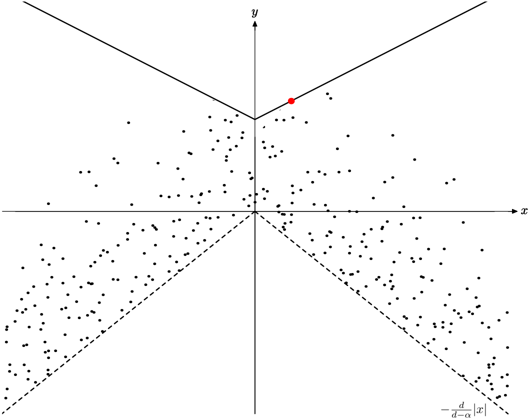

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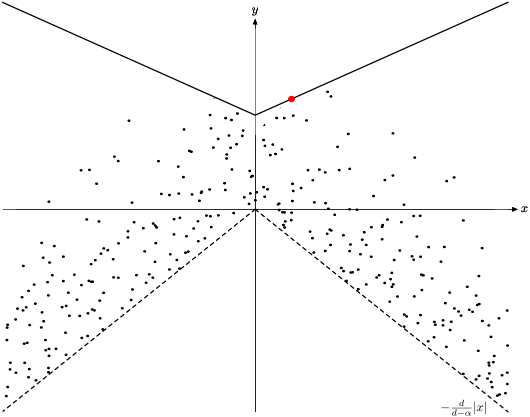

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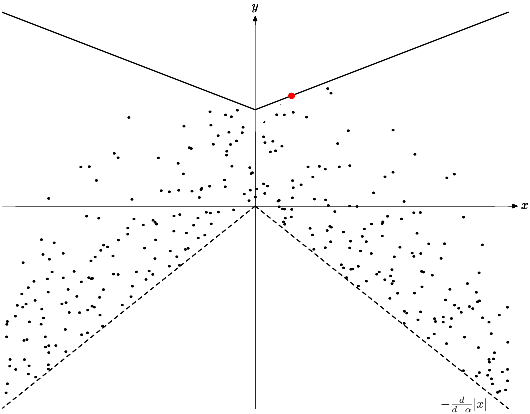

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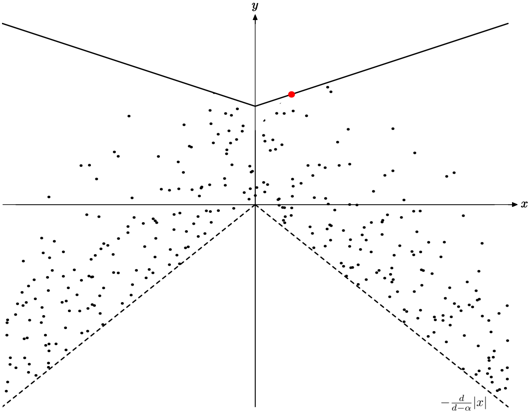

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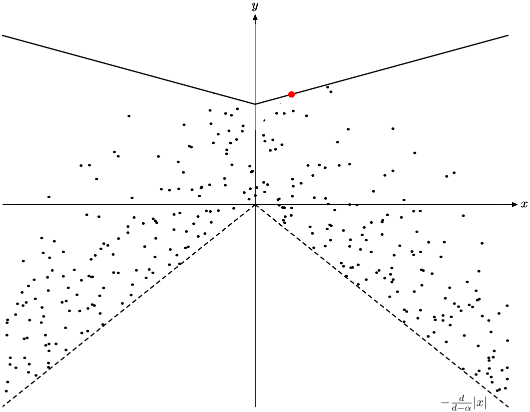

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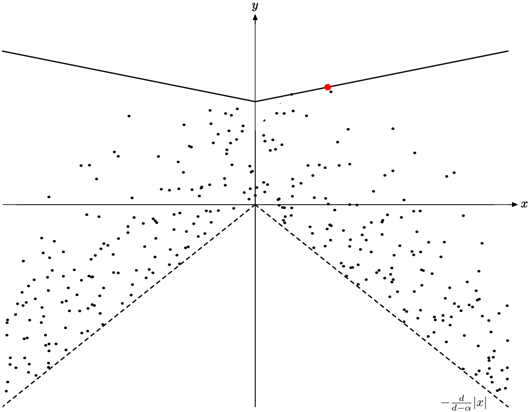

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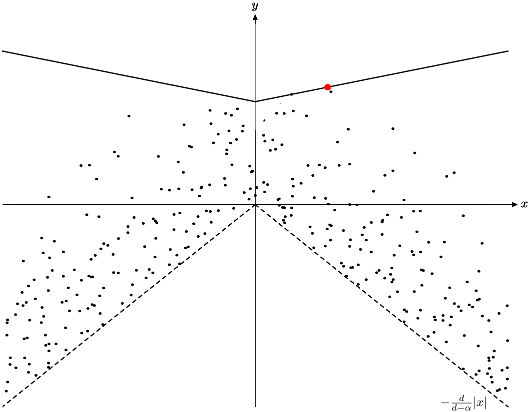

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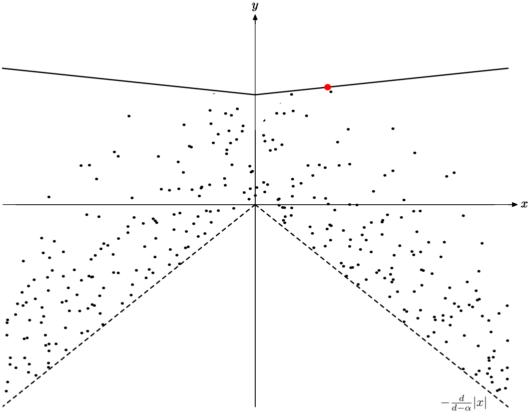

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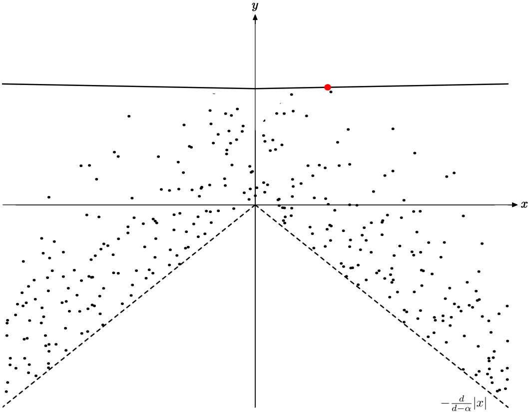

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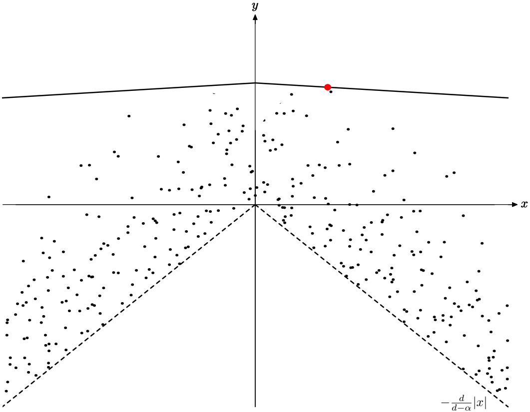

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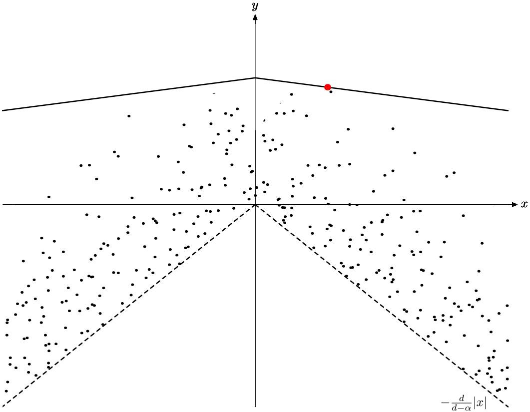

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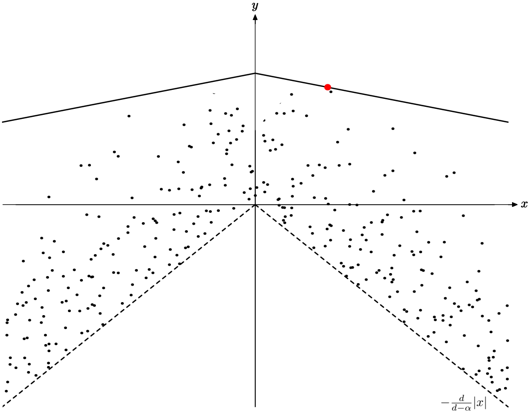

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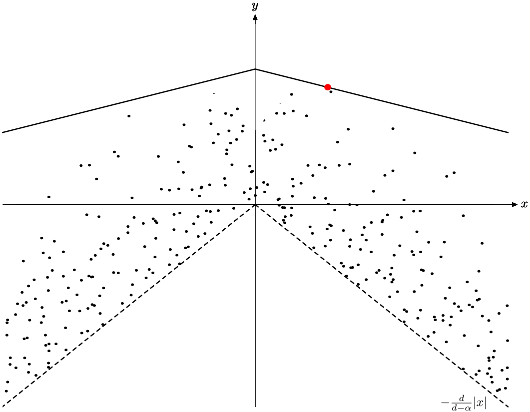

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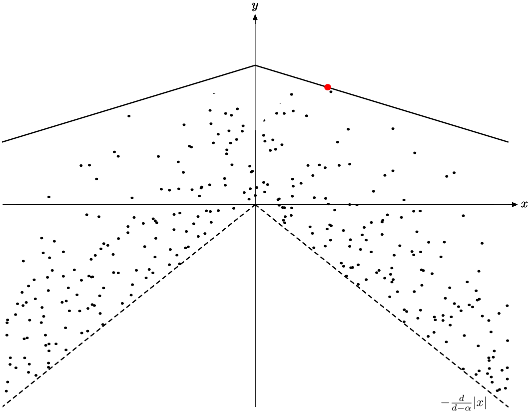

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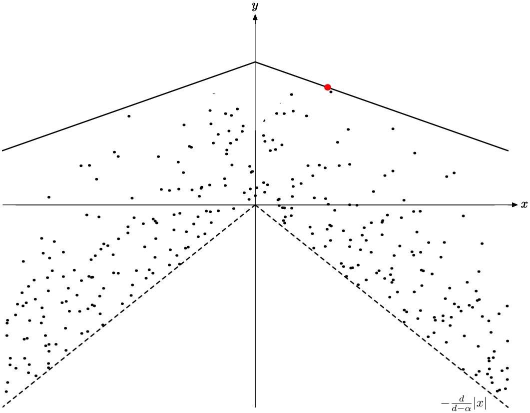

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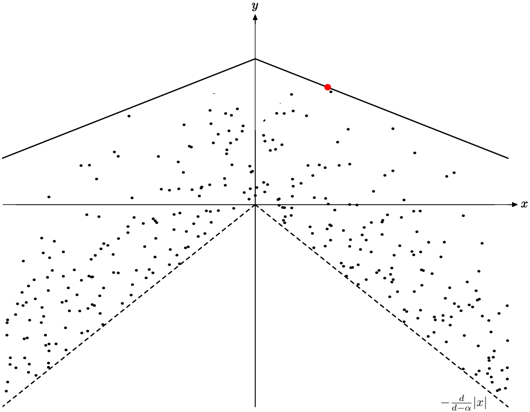

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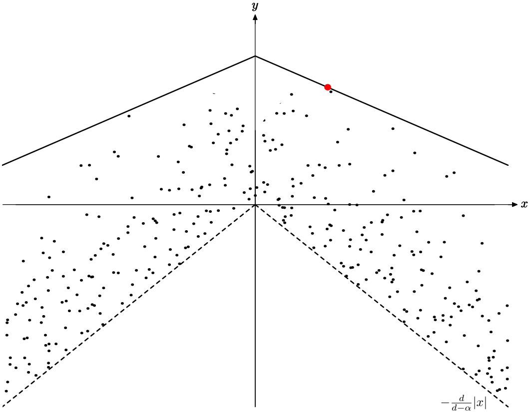

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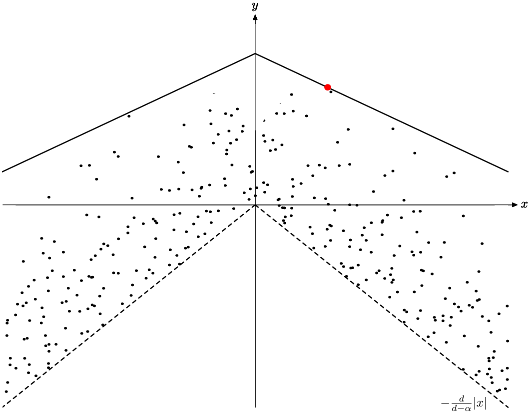

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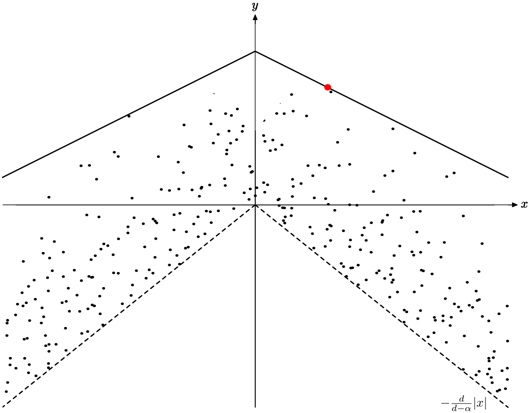

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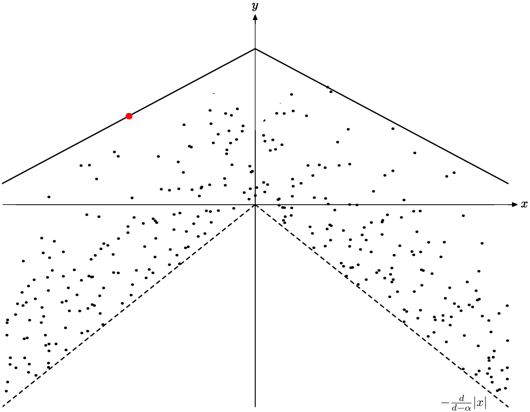

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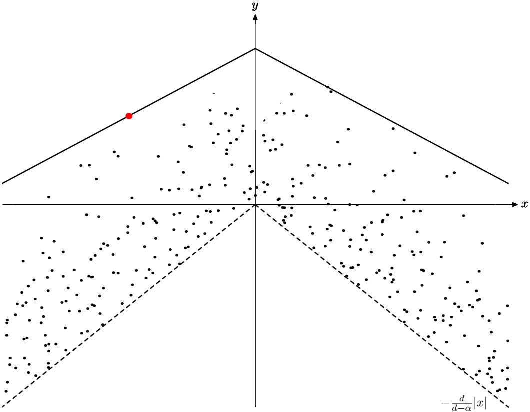

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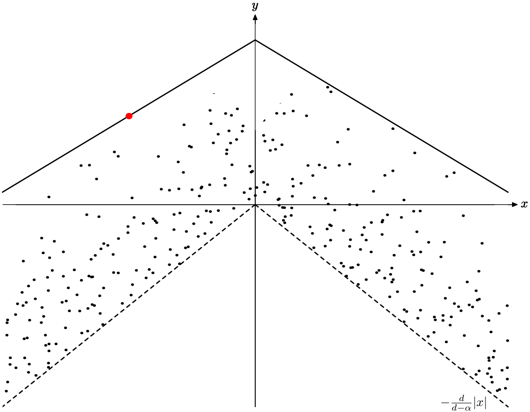

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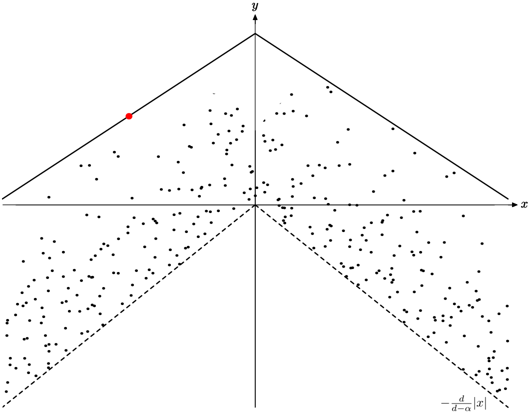

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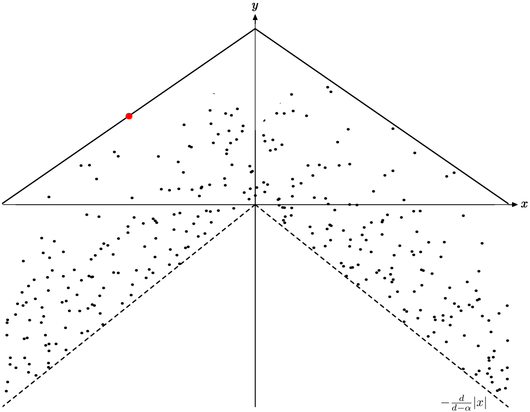

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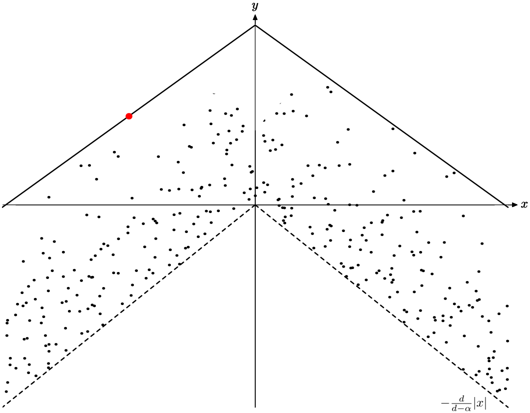

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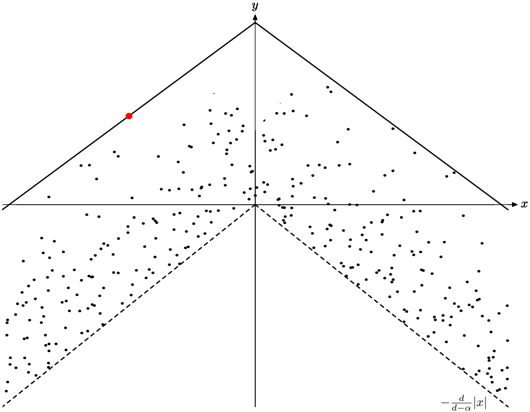

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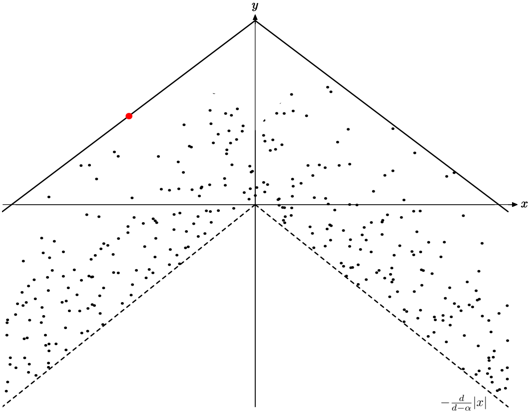

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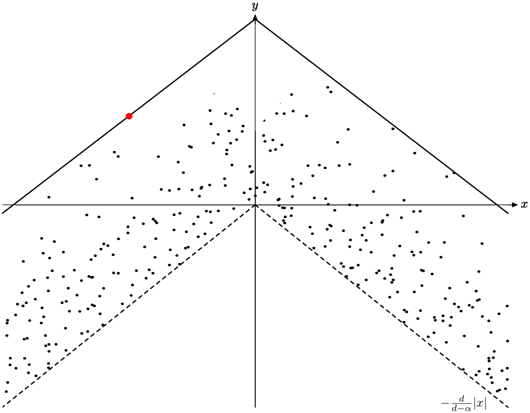

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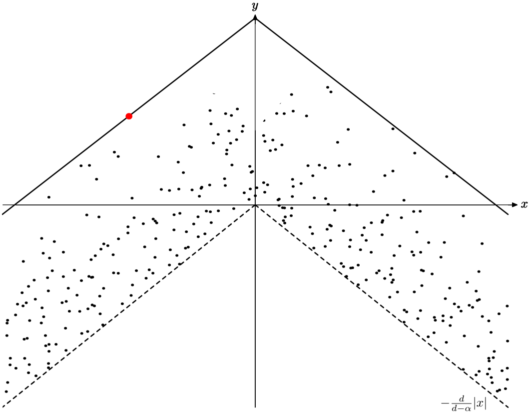

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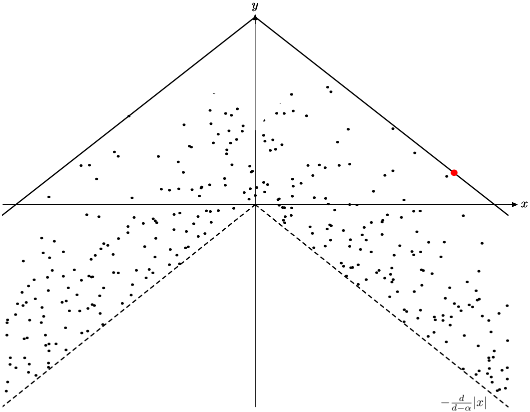

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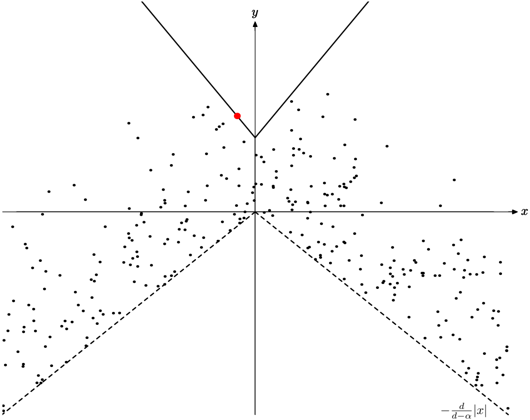
$$\Rightarrow \big(Y_s^{(1)}, Y_s^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{s}\big) \| Y_s^{(1)} \| \ : \ s>0 \big).$$

The second component corresponds to the second component of the tip of the cone that defines Y_s.



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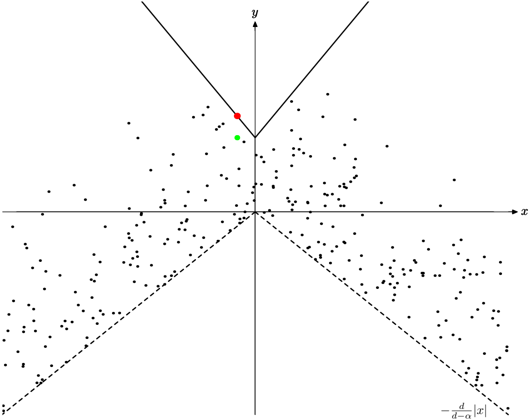

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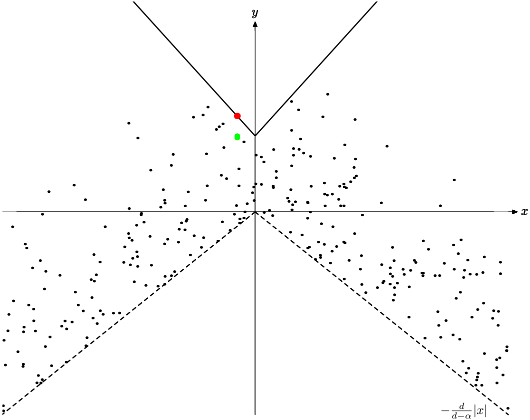
$$\Rightarrow \big(Y_s^{(1)}, Y_s^{(2)} + \tfrac{d}{\alpha-d}\big(1-\tfrac{1}{s}\big)\|Y_s^{(1)}\| \;:\; s>0 \big).$$

The second component corresponds to the second component of the tip of the cone that defines Y_s.



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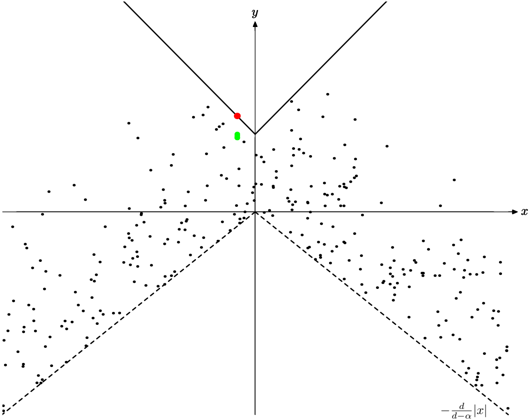

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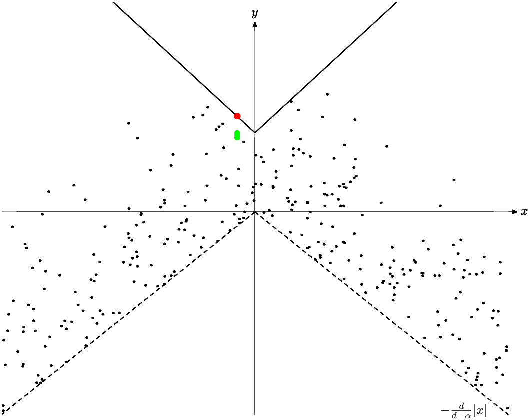

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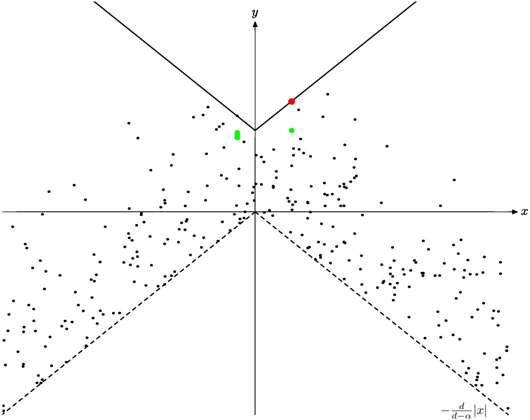

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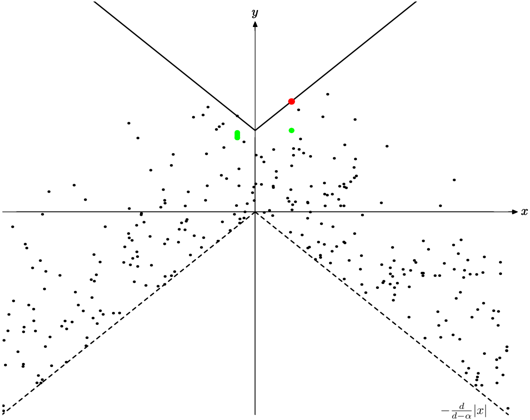

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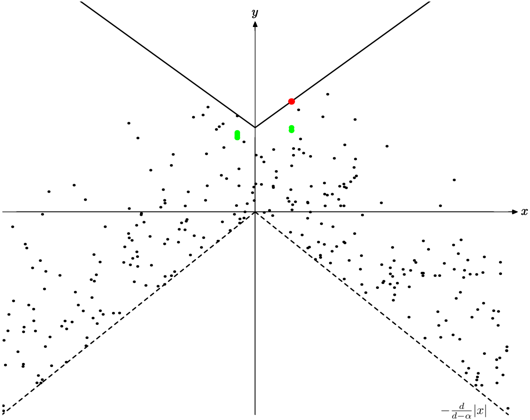

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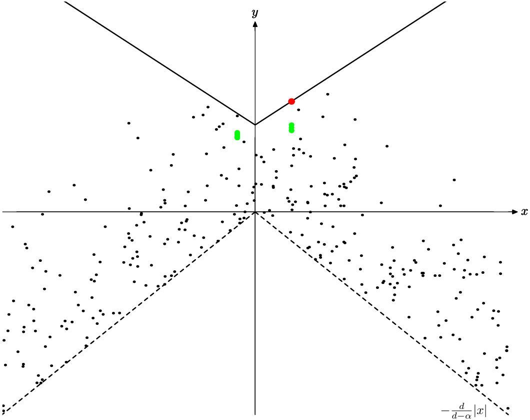

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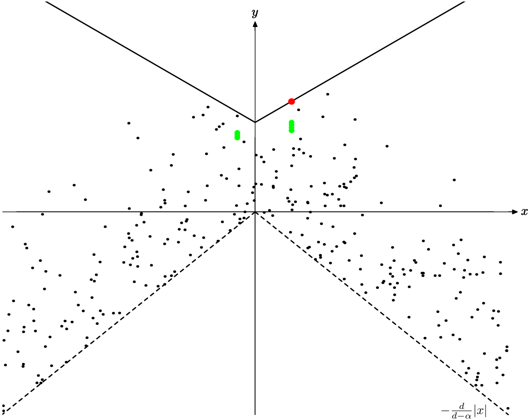

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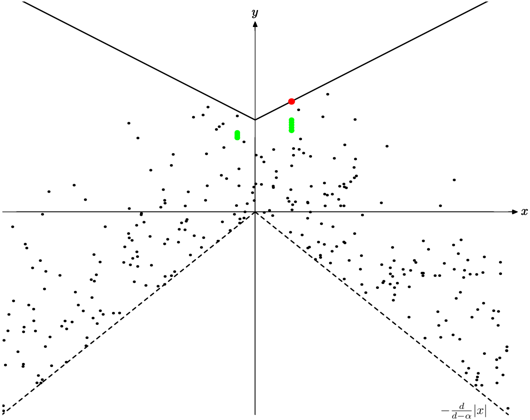

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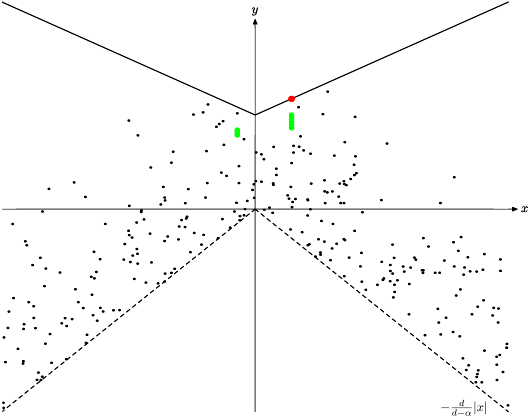

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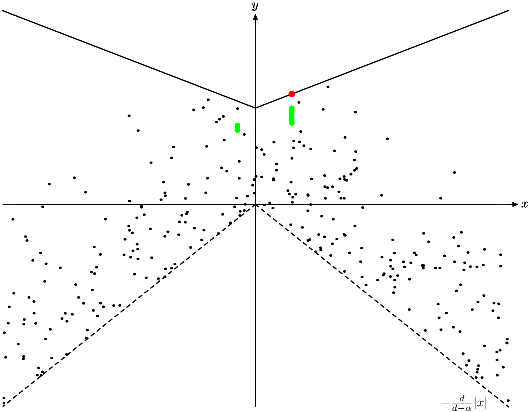

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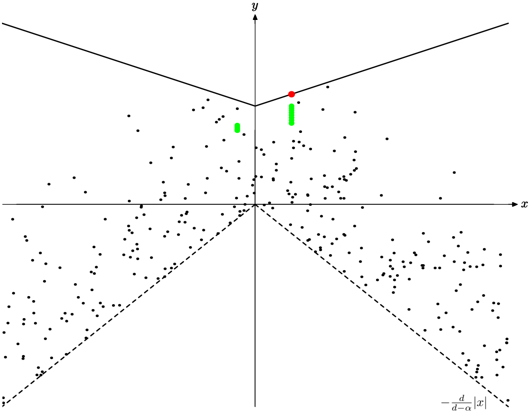

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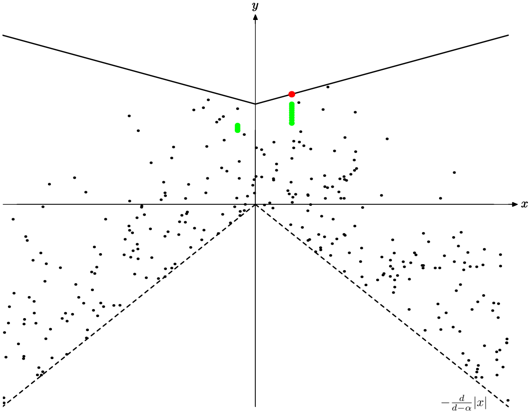

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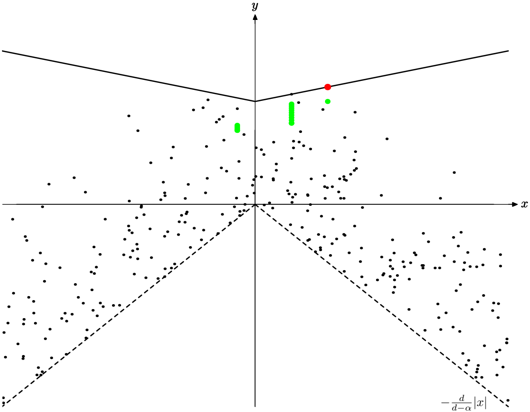

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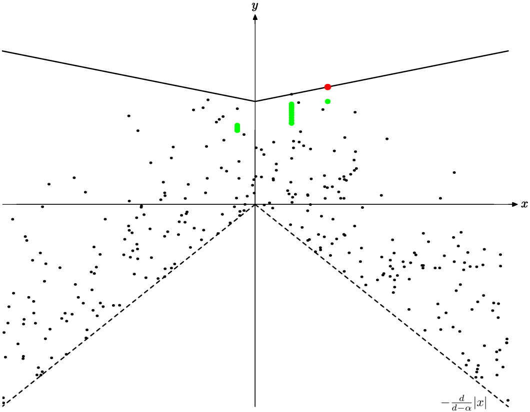

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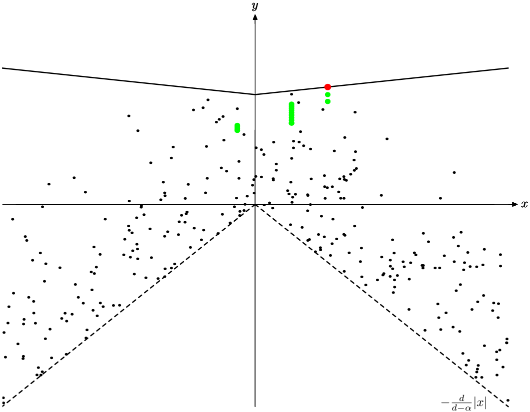

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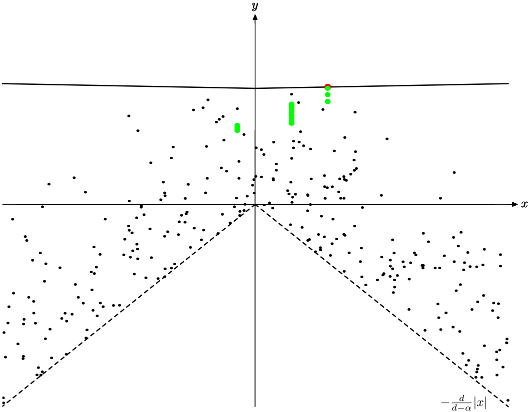

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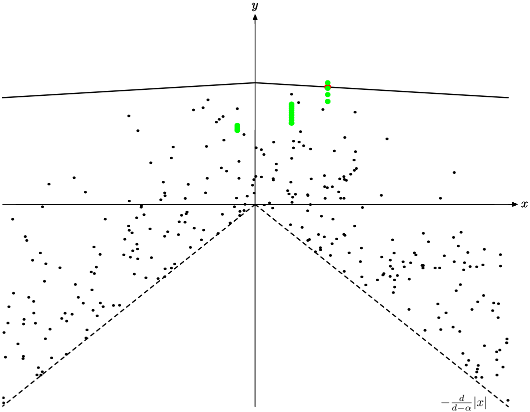

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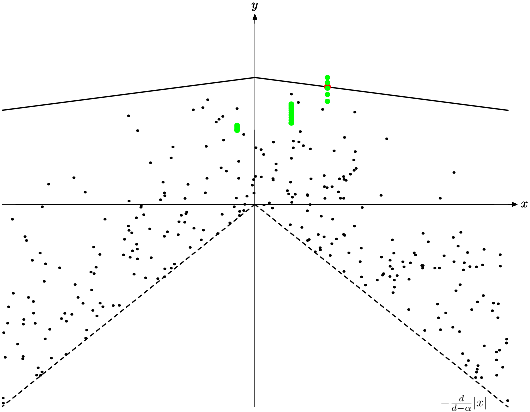

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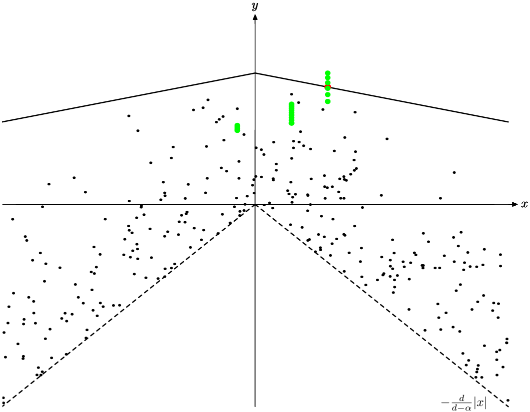

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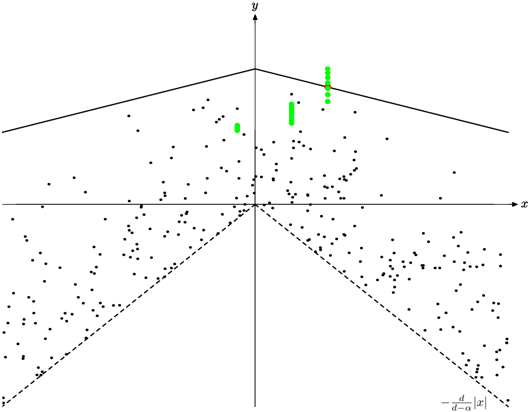

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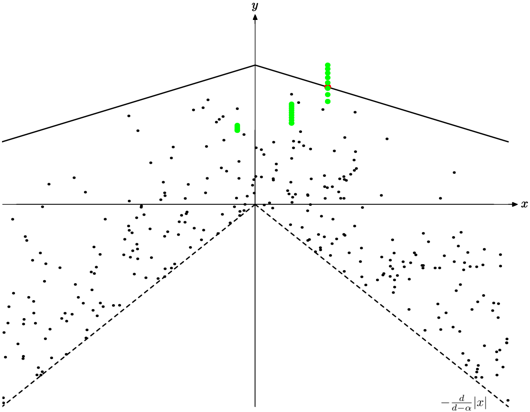

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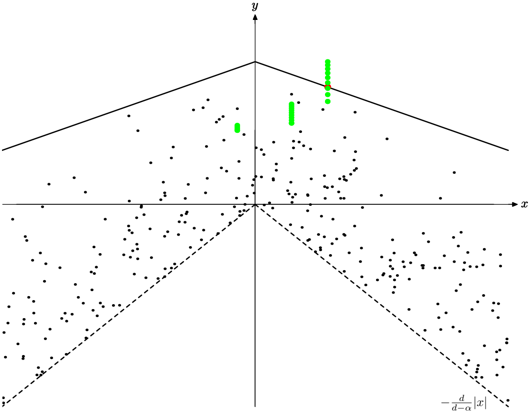

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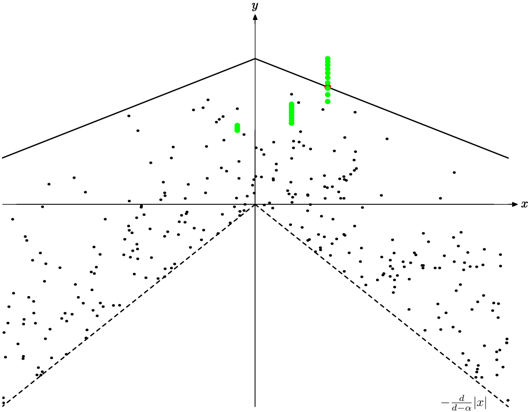

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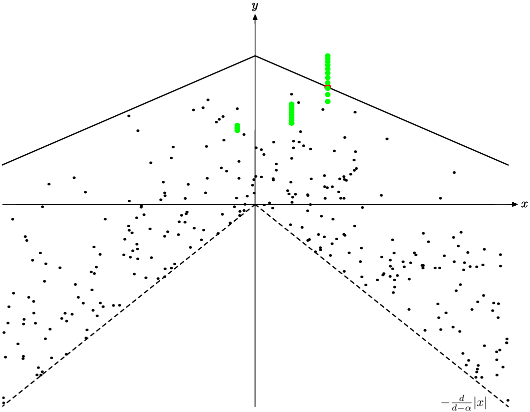

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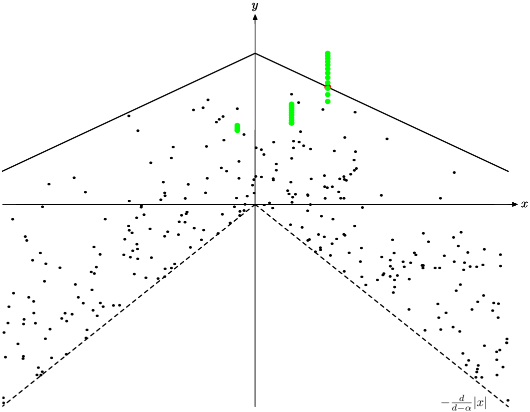

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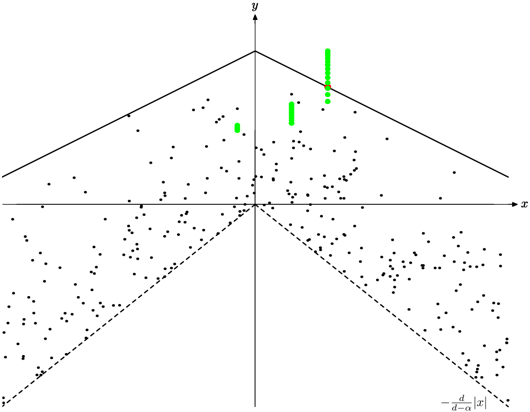

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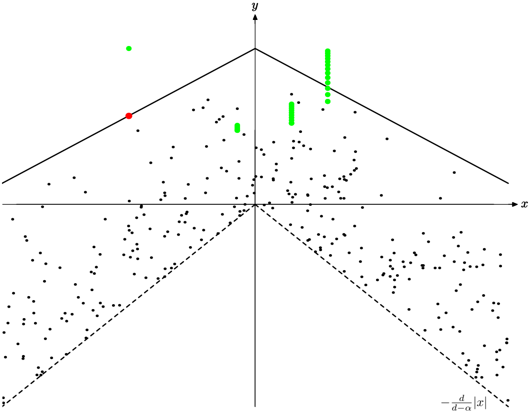

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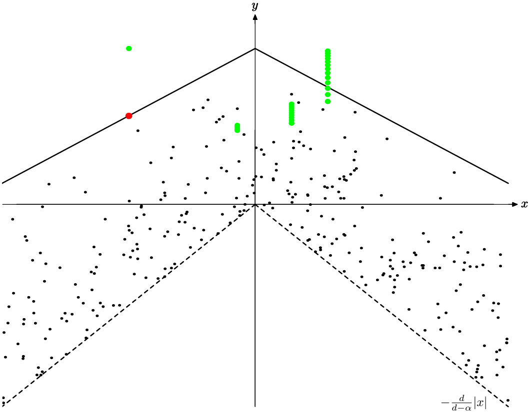

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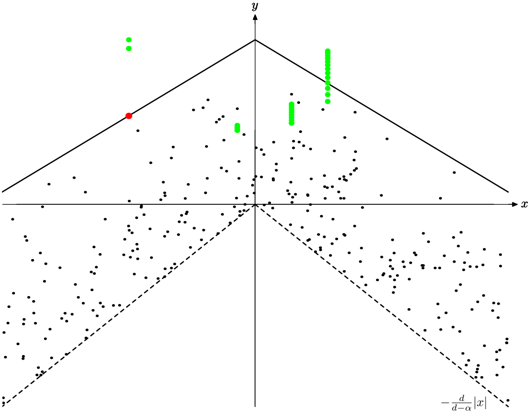

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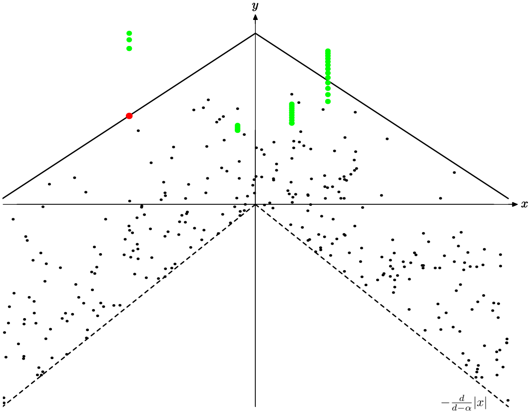

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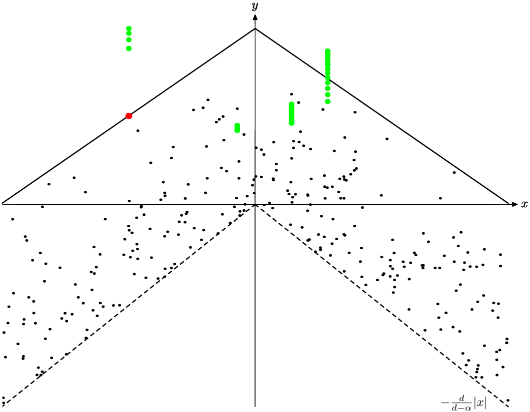

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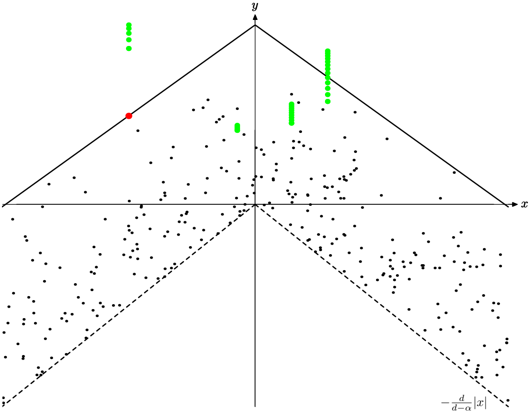

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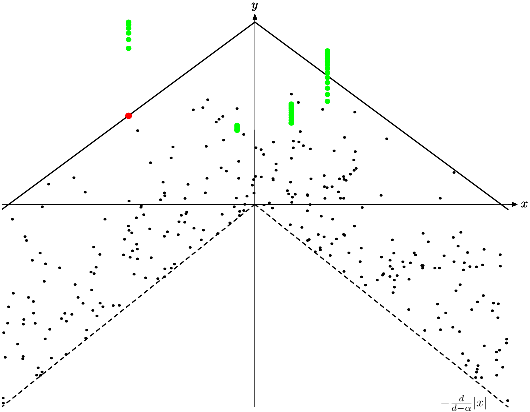

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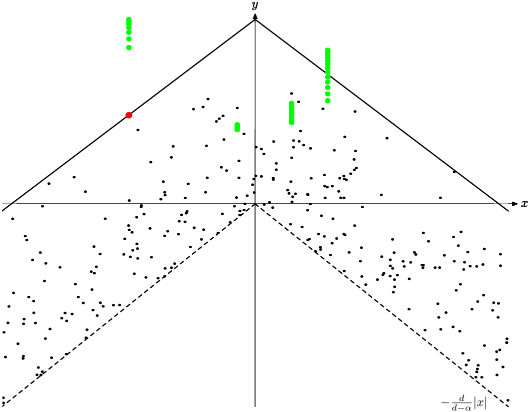

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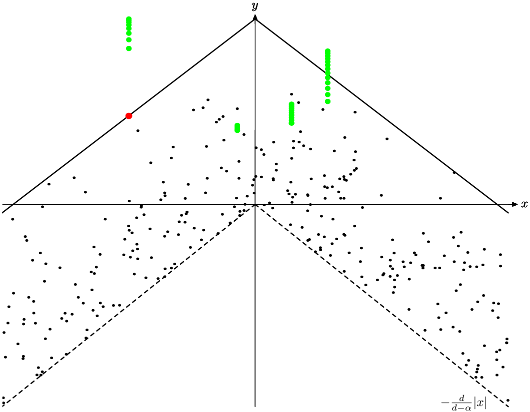

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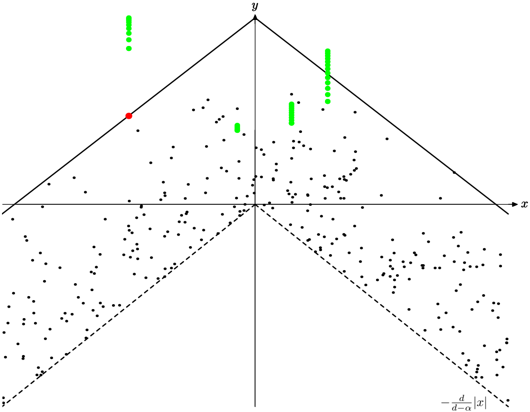

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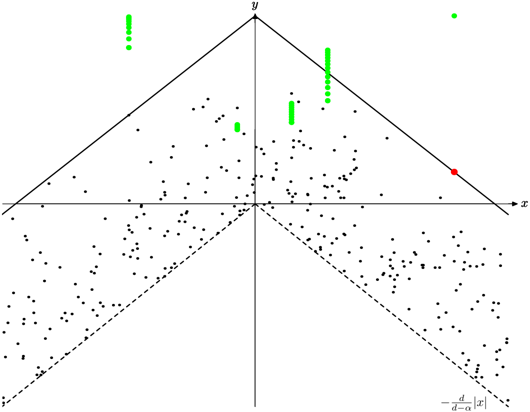

Almost sure behaviour

Recall our first theorem:

There exists a stochastic process $(Z_t \colon t > 0)$ with values in $\mathbb{Z}^d$ such that

$$\lim_{t\to\infty} \frac{u(t, Z_t)}{\sum_{z\in\mathbb{Z}^d} u(t, z)} = 1 \text{ in probability.}$$

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Question:

- How many sites are needed to support the bulk of the solution almost surely?

Two cities theorem

Theorem 3 (König, Lacoin, M, Sidorova 2007)

There exist two stochastic processes $(Z_t^{(1)} \colon t > 0)$ and $(Z_t^{(2)} \colon t > 0)$ with values in $\mathbb{Z}^d$ such that $Z_t^{(1)} \neq Z_t^{(2)}$ for all $t > 0$ and

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Remarks:

- At a typical large time the mass, which is thought of as a population, inhabits one site, interpreted as a city. At some rare times, however, word spreads that a better site has been found, and the entire population moves to the new city, so that at the transition times part of the population still lives in the old city, while part has already moved to the new one.

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- The term two cities theorem was suggested to us by S.A. Molchanov.

Two cities theorem: Key idea

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For a finer approximation we look at random walks which wander to a site z during the time interval $[0, \rho t]$ and stay there throughout $[\rho t, t]$. This has probability

$$\approx \exp\Big\{ - \|z\| \log \frac{\|z\|}{e\rho t} - 2dt + \eta(z) \Big\},$$

where $\eta(z) = \log \#\{$ paths of length $\|z\|$ from origin to $z\}$.

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where $\eta(z) = \log \#\{$ paths of length $\|z\|$ from origin to $z\}$. We obtain

$$\begin{aligned} \frac{1}{t} \log U(t) &\approx \sup_{z \in \mathbb{Z}^d} \sup_{\rho \in (0,1)} \Big\{ (1-\rho)\xi(z) - \frac{\|z\|}{t} \log \frac{\|z\|}{e\rho t} + \frac{\eta(z)}{t} \Big\} \\ &\approx \sup_{z \in \mathbb{Z}^d} \underbrace{\Big\{ \xi(z) - \frac{\|z\|}{t} \log \xi(z) + \frac{\eta(z)}{t} \Big\}}_{=: \Phi_t(z)}. \end{aligned}$$

Ageing

Roughly speaking, if a system exhibits ageing, the probability that there is no essential change of state between time t and time $t + s(t)$ is of constant order for a period $s(t)$ which depends increasingly, and often linearly, on the time t.

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Questions:

- Does the parabolic Anderson model exhibit ageing?
- How many time-scales are relevant to our model?

Ageing

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Let

$$v(t,x) = \frac{u(t,x)}{\sum_{z\in\mathbb{Z}^d} u(t,z)} \quad \text{for } t>0, x\in\mathbb{Z}^d.$$

Then there exists some $0<\theta(c)<1$ such that, for all $\epsilon>0$,

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}\Big\{ \sup_{x\in\mathbb{Z}^d} \big| v(t+ct,x) - v(t,x)\big| < \epsilon\Big\} \\ = \lim_{t\to\infty} \mathbb{P}\Big\{ \sup_{0\le s\le ct} \sup_{x\in\mathbb{Z}^d} \big| v(t+s,x) - v(t,x)\big| < \epsilon\Big\} \\ = \theta(c).\end{aligned}$$

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Remark: The limit $\theta(c)$ is not associated to a generalized arc-sine law, as typically observed in simple trap models, but a more complicated function.

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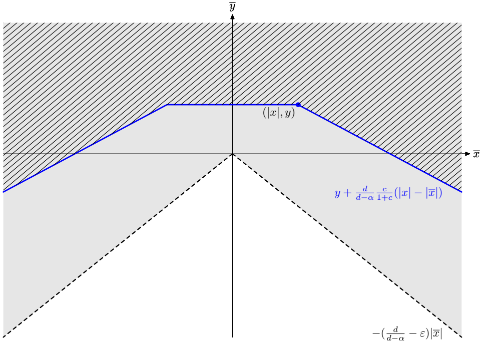

Summary

We have seen that for a potential with heavy tails the parabolic Anderson model shows interesting extreme behaviour, in particular

- the growth rate of the total mass is asymptotically random,
- the solution is asymptotically concentrated in a single point at most times,
- this point goes to infinity at superlinear speed,
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In the proofs we combine a very fine analysis of the random walk paths contributing in the Feynman-Kac formula with extreme value theory for the random field.

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