Intersection local time of Brownian motion: Tail behaviour and fractal applications

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Dedicated to S. James Taylor on the occasion of his 75th birthday.

We look at p independent Brownian motions $W_1, \ldots, W_p$ in $\mathbb{R}^d$ starting in the origin and each running for one time unit. By classical results of Dvoretzky, Erdős, Kakutani and Taylor, the intersection of the paths of these motions

$$S = \bigcap_{i=1}^{p} \big\{ x \in \mathbb{R}^d \,:\, x = W_i(t) \text{ for some } t \in [0,1] \big\}$$

contains points different from the starting point if and only if $p(d-2) < d$. In these cases the random set S of intersection points can be equipped with a natural finite measure ℓ, the intersection local time, which can be described symbolically by the formula

$$\ell(A) = \int_A dy \prod_{j=1}^{p} \int_0^1 ds\, \delta_y(W_j(s)), \text{ for } A \subset \mathbb{R}^d \text{ Borel.}$$

Focusing on two Brownian motions in the plane, we first describe the lower tail behaviour of ℓ and relate it to the intersection exponents $\xi_d(n,m)$ recently found by Lawler, Schramm and Werner in the case of dimension $d = 2$. The following result is from joint work with **Achim Klenke (Mainz)**.

Theorem: If U is an open set containing the origin, then

$$\lim_{\delta \downarrow 0} \frac{\log \mathbb{P}\big\{ \ell(U) < \delta \big\}}{\log \delta} = \frac{1}{2}\, \xi_2(2,2) = \frac{35}{24}.$$

In the same project we use this result to derive the *multifractal spectrum* of the intersection local times.

Theorem: For all $2 \le a \le \frac{70}{11}$, almost surely,

$$\dim \Big\{ x \in S \,:\, \limsup_{r \downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a \Big\} = \frac{1}{12}\Big(\frac{70}{a} - 11 \Big).$$

For all other values of a the set above is empty, almost surely. All these results have analogues for more then two Brownian motions and in the case $d = 3$.

In an ongoing project with **Narn-Rueih Shieh (Taipei)** we study the question, raised by S.J. Taylor in the 1980s, of the exact packing measure of the set D of double points of a Brownian motion in $\mathbb{R}^3$.

Theorem: Let ϕ be any gauge function such that

$$\int_{0+} r^{-1-\xi_3(2,2)}\phi(r)^{\xi_3(2,2)}\,dr = \infty.$$

Then, almost surely, $\mathcal{P}^\phi(D) = \infty$.

We conjecture that this result is sharp and present partial results in this direction. These results ensure in particular that for $\phi(r) = r\log(1/r)^\alpha$, almost surely,

$$\mathcal{P}^\phi(D) = \begin{cases} \infty & \text{if } \alpha \geq -1/\xi_3(2,2), \\ 0 & \text{if } \alpha < -1/\xi_3(2,2). \end{cases}$$

To round up, *upper* tails of intersection local times were studied in a joint project with **Wolfgang König (Leipzig)**. We state the results for two Brownian motions in $\mathbb{R}^3$ running for unbounded time.

Theorem: If U is a bounded open set containing the origin, then

$$\lim_{a\to\infty} \frac{1}{\sqrt{a}} \log \mathbb{P}\big\{\ell(U) > a\big\} = -\theta(U),$$

for a constant $\theta(U)$ given by

$$\theta(U) = \inf\Big\{ \|\nabla\psi\|_2^2 \,:\, \psi \in C_0^2(\mathbb{R}^d),\, \int_U \psi^4 = 1\Big\}.$$

As the upper tails decay on an exponential rather than polynomial scale, the multifractal spectrum is trivial at the thick end. However in the same project we obtain the following *logarithmic* multifractal spectrum, or spectrum of thick points.

Theorem: For all $0 \leq a \leq 1/\theta(B(0,1))^2$, almost surely,

$$\dim\Big\{ x \in S \,:\, \limsup_{r\downarrow 0} \frac{\ell(B(x,r))}{r[\log(1/r)]^2} = a \Big\} = 1 - \sqrt{a}\theta(B(0,1))\,.$$

For all other values of a the set above is empty, almost surely.

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