Multifractal analysis of branching measure on a Galton-Watson tree

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Abstract

Suppose that μ is the branching measure on the boundary of a supercritical Galton-Watson tree with offspring variable N such that $E[N \log N] < \infty$. We survey recent results on the multifractal spectrum and logarithmic multifractal spectrum of the measure μ, and also add a new result. Our tool is percolation on the Galton-Watson tree.

Keywords: Galton-Watson tree, branching measure, Hausdorff dimension, thin point, thick point, dimension spectrum, multifractal spectrum.

1 Branching measure on a Galton-Watson tree

We consider a Galton–Watson tree T starting from a single progenitor, the root ρ. Let N be its offspring variable and suppose it satisfies $\mathfrak{m} := EN > 1$, which means that the process is supercritical. A sequence $V = (v_0, v_1, \dots)$ of vertices in T such that $v_0 = \rho$ and v_{i+1} is a child of v_i, for every i, is called a *ray*. By ∂T we denote the *boundary* of the tree consisting of all rays. We say that the tree survives if $\partial T \neq \emptyset$ and denote by $q > 0$ the survival probability. The set ∂T can be equipped with a metric d by letting $d(U, V) = e^{-n}$ if n is the generation of the last, or oldest, vertex present in both rays U and V. With respect to this metric it is well–known that, almost surely on

$\partial T \neq \emptyset$, the Hausdorff dimension of ∂T is $\mathfrak{a} := \log \mathfrak{m}$, see Hawkes (1981) and Lyons (1990), and for example Liu (2001), Watanabe (2004) for refinements.

Suppose that Z_n is the number of individuals in the n^{th} generation of the tree. Then $W_n := Z_n/\mathfrak{m}^n$ defines a martingale, and hence the limit

$$W = \lim_{n\to\infty} \frac{Z_n}{\mathfrak{m}^n}$$

exists almost surely. By a famous theorem of Kesten and Stigum (1966) we have

$$P\{W > 0\} = \begin{cases} q & \text{if } E[N \log N] < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We are only interested in the case $E[N \log N] < \infty$. In this case, for every vertex $v \in T$ we let $T(v)$ be the subtree consisting of all successors of v, and by $W(v)$ the martingale limit of $T(v)$. This allows us to build a natural measure μ, called the *branching measure*, on ∂T by the requirement

$$\mu\{V \in \partial T \,:\, v \in V\} = \frac{W(v)}{\mathfrak{m}^n} \qquad \text{for all vertices } v \in T \text{ in generation } n.$$

Note that the set on the left is also the closed ball of radius e^{-n} around any ray $(v_0, v_1, \ldots)$ with $v_n = v$. Almost surely, for μ-almost every $V \in \partial T$,

$$\lim_{n\to\infty} \frac{\log \mu(B(V, e^{-n}))}{-n} = \mathfrak{a},$$

i.e. the *local dimension* of μ is $\mathfrak{a}$ almost everywhere. We are interested in the question whether there exist exceptional rays, where the local dimension has a different value. In this case the measure μ is called *multifractal*. If the measure is multifractal, we ask how many rays (in the sense of Hausdorff dimension) exist for a given local dimension, the resulting function is called the *multifractal spectrum*.

From now on we assume that $P\{N = 0\} = 0$. This is no loss of generality, indeed, if we prune a Galton-Watson tree by removing all its finite subtrees, we do not change its boundary. The pruned tree is still a Galton-Watson tree (with a modified offspring variable), see Athreya and Ney (1972, Chapter 1, Section 12). To avoid trivialities we also assume $P\{N = k\} < 1$ for all positive integers k.

2 The multifractal spectrum

We say that the offspring variable is of *Schröder type* if $P\{N \leq 1\} > 0$ and otherwise, if $P\{N \leq 1\} = 0$, we say it is of *Böttcher type*.

Theorem 2.1 (Multifractal spectrum I, thin part)
Suppose N is of Schröder type. Define $\tau := -\log P\{N=1\}/\mathfrak{a}$. Then, for all $\mathfrak{a} \le \theta \le \mathfrak{a}\big(1+\frac{1}{\tau}\big)$, almost surely,

$$\dim \Big\{ V \in \partial T \,:\, \limsup_{n\to\infty} \frac{\log \mu B(V, e^{-n})}{-n} = \theta \Big\} = \mathfrak{a} \left(\frac{\mathfrak{a}}{\theta} (1+\tau) - \tau \right). \qquad (1)$$

If $\theta > \mathfrak{a}\big(1+\frac{1}{\tau}\big)$, almost surely, the set on the left hand side of (1) is empty, however if $\theta = \mathfrak{a}\big(1+\frac{1}{\tau}\big)$ it is almost surely nonempty.

This result was first stated and partially proved by Shieh and Taylor (2002). A full proof is given in Mörters and Shieh (2004).

If N is 'heavy tailed' there is also a nontrivial multifractal spectrum for local dimensions below the typical value. For the formulation of this result, let

$$r := \liminf_{x\uparrow\infty} \frac{-\log P\{N > x\}}{\log x}.$$

Note that $r \in [1,\infty]$ and $r = \infty$ if and only if all positive integer moments of N exist. We use the convention $1/\infty = 0$.

Theorem 2.2 (Multifractal spectrum II, thick part)
For all $\mathfrak{a}\big(1-\frac{1}{r}\big) \le \theta \le \mathfrak{a}$, almost surely,

$$\dim \Big\{ V \in \partial T \,:\, \liminf_{n\to\infty} \frac{\log \mu B(V, e^{-n})}{-n} = \theta \Big\} = \mathfrak{a}\,(1-r) + \theta r\,. \qquad (2)$$

If $\theta < \mathfrak{a}\big(1-\frac{1}{r}\big)$, almost surely, the set on the left hand side of (2) is empty, however if $\theta = \mathfrak{a}\big(1-\frac{1}{r}\big)$ it is almost surely nonempty.

This result substantiates Remark 5.1 in Shieh and Taylor (2002); the result is yet unpublished. Its proof, which we will sketch in Section 4, is a straightforward adaptation of the method of Mörters and Shieh (2002).

3 The logarithmic multifractal spectrum

From Theorems 2.1 and 2.2 we infer that, if N is of Böttcher type and not heavy tailed (i.e. $r = \infty$), then the branching measure is not multifractal; see also Liu (2001, Theorem 4.1). In order to obtain nontrivial spectra at both ends we study variations on a finer scale. This idea goes back to Shieh and Taylor (1998) who use the term *logarithmic multifractal analysis*.

We first look at points which are exceptionally thin. We let $M_- = \operatorname{essinf} N$ and $\lambda_- = 1 - \log \mathfrak{m}/\log M_-$. Under our standing assumptions we always have $\lambda_- < 0$. In the Böttcher case the value

$$r_- := \liminf_{x \downarrow 0} \frac{-\log P\{W < x\}}{x^{1/\lambda_-}}$$

is positive and finite.

Theorem 3.1 (Spectrum of thin points) *In the Böttcher case we have, almost surely,*

$$\dim \left\{ V \in \partial T \,:\, \liminf_{n\to\infty} \frac{\mu B(V, e^{-n})}{\mathfrak{m}^{-n} n^{\lambda_-}} = \theta \right\} = \mathfrak{a} - r_- \theta^{1/\lambda_-}, \qquad (3)$$

for all $\theta \geq (\mathfrak{a}/r_-)^{\lambda_-}$. If $\theta < (\mathfrak{a}/r_-)^{\lambda_-}$, almost surely, the set on the left hand side of (3) is empty, however if $\theta = (\mathfrak{a}/r_-)^{\lambda_-}$ it is almost surely nonempty.

This result is proved in Mörters and Shieh (2002).

We now turn to the study of exceptionally thick points. Let $M_+ = \operatorname{esssup} N$ and $\lambda_+ = 1 - \log \mathfrak{m}/\log M_+$. If $M_+ = \infty$ this means that $\lambda_+ = 1$. Under our standing assumptions we always have $\lambda_+ > 0$. If (and only if) ess sup $N < \infty$ or $0 < \sup\{t \,:\, E[\exp(tN)] < \infty\} < \infty$, the value

$$r_+ := \liminf_{x \uparrow \infty} \frac{-\log P\{W > x\}}{x^{1/\lambda_+}}$$

is positive and finite.

Theorem 3.2 (Spectrum of thick points) *If either ess sup $N < \infty$ or $0 < \sup\{t \,:\, E[\exp(tN)] < \infty\} < \infty$ we have, almost surely,*

$$\dim \left\{ V \in \partial T \,:\, \limsup_{n\to\infty} \frac{\mu B(V, e^{-n})}{\mathfrak{m}^{-n} n^{\lambda_+}} = \theta \right\} = \mathfrak{a} - r_+ \theta^{1/\lambda_+}, \qquad (4)$$

for all $0 \leq \theta \leq (\mathfrak{a}/r_+)^{\lambda_+}$. If $\theta > (\mathfrak{a}/r_+)^{\lambda_+}$, almost surely, the set on the left hand side of (4) is empty, however if $\theta = (\mathfrak{a}/r_+)^{\lambda_+}$ it is almost surely nonempty.

This result is proved in Mörters and Shieh (2002), a partial result can be found in Shieh and Taylor (2002).

Remark: The thin and thick points of this paper have exceptionally small or large mass in a sequence of centred balls with radii decreasing to zero. An entirely different problem would be to study points which have exceptional behaviour in *all* small centred balls. This problem remains open.

4 Sketch of the proof of Theorem 2.2.

We use percolation on trees to study Hausdorff dimension, an idea suggested by Lyons (1990). Our method is also influenced by the techniques of Khoshnevisan, Peres and Xiao (2000) in the Euclidean case.

If T is any tree and $p \in [0,1]$, we attach to each edge e of the tree an independent $\{0,1\}$-valued random variable $X(e)$ with $P\{X(e)=1\}=p$. We denote by T^* the connected component of the root ρ in the graph consisting of all edges e with $X(e)=1$. We say that T^* is the result of running *percolation with retention parameter* p on T. The following lemma is due to Lyons (1990), see also Remark 2 in Mörters and Shieh (2004).

Lemma 4.1 *For an analytic set* $S \subset \partial T$*, if* $p < \exp(-\dim S)$*, then* $S \cap \partial T^* = \emptyset$ *almost surely, and if* $p > \exp(-\dim S)$*, then* $\partial T^* \cap S \neq \emptyset$ *with positive probability.*

If we run independent percolation on a Galton-Watson tree T, the unconditional distribution of T^* is again the law of a Galton-Watson tree. If $\mathfrak{m}$ is the mean offspring number of T, then $p\mathfrak{m}$ is the mean offspring number of T^*. Hence $\partial T^* \neq \emptyset$ with positive probability if and only if $p > 1/\mathfrak{m}$.

Lemma 4.2 *For any retention parameter* $1/\mathfrak{m} < p \leq 1$*,*

$$\lim_{x\uparrow\infty} \frac{1}{\log x} \log P\big(\{W > x\} \cap \{\partial T^* \neq \emptyset\}\big) = -r\,.$$

Remark: In the case $p = 1$ this is the well-known, see for example Liu (2001, p.202). Intuitively it is clear that the events $\{W > x\}$ and $\{\partial T^* \neq \emptyset\}$ are nonnegatively correlated. The proof can achieved with the technique of Lemma 3.4 in Mörters and Shieh (2002). ■

For $\theta \geq \mathfrak{a}$ define an analytic subset of ∂T by

$$A(\theta) = \Big\{ V \in \partial T \,:\, \liminf_{n\to\infty} \tfrac{-1}{n} \log \mu B(V, e^{-n}) \leq \theta \Big\}.$$

We exploit Lemma 4.2 in order to determine the probability that a ray in $A(\theta)$ survives percolation.

Lemma 4.3

(a) If the retention parameter is chosen satisfying

$$\frac{1}{\mathfrak{m}} < p < \frac{1}{\mathfrak{m}} \exp\big(r(\mathfrak{a} - \theta)\big),$$

then $A(\theta) \cap \partial T^* = \emptyset$ almost surely.

(b) If the retention parameter is chosen satisfying

$$p \geq \frac{1}{\mathfrak{m}} \exp\big(r(\mathfrak{a} - \theta)\big),$$

then $A(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$.

Before proving this, we show how Theorem 2.2 follows from this. Note first that the *upper bound* in (2) follows readily by combining Lemma 4.1 with Lemma 4.3 (a).

To prove the *lower bound* in (2) we fix $\theta \geq \mathfrak{a}$ and study the analytic set

$$S(\theta) = \Big\{ V \in \partial T \,:\, \liminf_{n\to\infty} \tfrac{-1}{n} \log \mu B(V, e^{-n}) = \theta \Big\}.$$

We run percolation with retention parameter

$$p = \frac{1}{\mathfrak{m}} \exp\big(r(\mathfrak{a} - \theta)\big)$$

and note that

$$S(\theta) \cap \partial T^* = (A(\theta) \cap \partial T^*) \setminus \bigcap_{N=1}^{\infty} A(\theta - \tfrac{1}{N}).$$

Now $A(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$, by Lemma 4.3 (b). On the other hand, by Lemma 4.3 (a) we know that $A(\theta - \frac{1}{N}) \cap \partial T^* = \emptyset$ almost surely. We infer that, with positive probability, $S(\theta) \cap \partial T^* \neq \emptyset$. Now, by Lemma 4.1, the Hausdorff dimension of $S(\theta)$ is bounded below by $-\log p = \mathfrak{a}(1-r)+\theta r$ with positive probability. The complementary property $\dim S(\theta) < \mathfrak{a}(1-r)+\theta r$ is a property of a tree T, which is inherited by all its subtrees $T(v)$ and also holds for all finite trees, hence it must have probability zero or one. Therefore our lower bound holds with probability one, and this finishes the proof of the lower bound in (2). ■

Proof of Lemma 4.3. Assume first that p is chosen as in part (a) of the lemma. Note that, for any n and $\varepsilon > 0$, if $A(\theta)\cap\partial T^* \neq \emptyset$ there exists a vertex $v \in T^*$ with $|v| =: m \geq n$ such that $\partial T(v)^* \neq \emptyset$ and $-\log\mu(\partial T(v)) \leq m(\theta+\varepsilon)$. Hence, we obtain, using Lemma 4.2,

$$\begin{aligned}
&P\{A(\theta)\cap\partial T^*\neq\emptyset\}\\
&\quad\leq \sum_{m=n}^{\infty} E \sum_{\substack{v\in T^*\\ |v|=m}} P\big\{-\log\mu(\partial T(v))\leq m(\theta+\varepsilon) \text{ and } \partial T(v)^*\neq\emptyset \,\big|\, v\in T^*\big\}\\
&\quad= \sum_{m=n}^{\infty} E \sum_{\substack{v\in T^*\\ |v|=m}} P\big\{W\geq \exp(m(\log\mathfrak{m}-\theta-\varepsilon)) \text{ and } \partial T^*\neq\emptyset\big\}\\
&\quad\leq \sum_{m=n}^{\infty} p^m\mathfrak{m}^m \exp\big(-rm(\mathfrak{a}-\theta-\varepsilon)\big)\\
&\quad= \sum_{m=n}^{\infty} \exp\big(m\big(\log p+\mathfrak{a}-r(\mathfrak{a}-\theta-\varepsilon\,)\big),
\end{aligned}$$

recalling that $\mathfrak{a} = \log\mathfrak{m}$. By our choice of p, we may choose $\varepsilon > 0$ so small that the sum on the right is convergent, and hence the upper bound converges to zero as $n\to\infty$.

Now assume that p is chosen as in part (b) of the lemma. For every positive integer N we let

$$U(N) = \Big\{V\in\partial T \;:\; \text{there is } n\geq N \text{ with } -\log\mu B(V,e^{-n}) < n(\theta+\tfrac{1}{N})\Big\}.$$

Our aim is to show that $U(N)\cap\partial T^*$ is dense in ∂T^* almost surely. But we first argue how we can use this to finish the proof of Lemma 4.3 (b). Note that $U(N)\cap\partial T^*$ is relatively open in ∂T^*, which is a compact, hence complete, metric space. By *Baire's Theorem*,

$$\bigcap_{N=1}^{\infty} U(N)\cap\partial T^* \text{ is dense in } \partial T^*.$$

But note that $\bigcap_{N=1}^{\infty} U(N) = A(\theta)$, hence $A(\theta) \cap \partial T^*$ must be dense in ∂T^* almost surely. This implies that $A(\theta) \cap \partial T^*$ is nonempty almost surely conditional on $\partial T^* \neq \emptyset$, which is the required statement.

To show that, for fixed integer N, the set $U(N) \cap \partial T^*$ is dense in ∂T^* we fix some vertex $v \in T$, denote its generation by m, and show that, almost surely conditional on $\partial T(v)^* \neq \emptyset$, we have $U(N) \cap \partial T(v)^* \neq \emptyset$.

Let $\mathcal{G}(n)$ be the σ-field generated by the finite subtree $T_n(v) \subset T(v)$, consisting of the first n generations of $T(v)$, together with the random variables $\{X(e) \colon e \in T_n(v)\}$. Let $\mathcal{K}_n$ be the collection of vertices in $T(v)^*$ of generation n and let K_n be the cardinality of $\mathcal{K}_n$. Note that the random variable K_n is $\mathcal{G}(n)$-measurable. We let $n_1, n_2, \ldots$ be a sequence of positive deterministic integers to be determined later, and define a sequence of $\mathcal{G}(n)$-stopping times $N_1, N_2, \ldots$ by $N_0 = 1$ and, for $k \geq 1$,

$$N_k = \min\Big\{ n > N_{k-1} + n_{k-1} \,:\, K_n \geq \exp\Big(n\big(r(\mathfrak{a} - \theta) - \tfrac{1}{k}\big)\Big)\Big\}.$$

Almost surely on $\partial T(v)^* \neq \emptyset$, we have

$$\dim \partial T(v)^* = \log(p\,\mathfrak{m}) = r(\mathfrak{a} - \theta).$$

This implies that $\{N_j \,:\, j \geq 1\}$ is a sequence of finite stopping times, almost surely on $\partial T(v)^* \neq \emptyset$. For every vertex $w \in T(v)$ of generation N_j define the event

$$\begin{aligned} E(w) \;&:=\; \Big\{ -\log \mu(\partial T(w)) < (N_j + m)(\theta + \tfrac{1}{N}) \text{ and } \partial T(w)^* \neq \emptyset \Big\} \\ &=\; \Big\{ W(w) > \exp\big((N_j + m)\big(\mathfrak{a} - \theta - \tfrac{1}{N}\big)\big) \text{ and } \partial T(w)^* \neq \emptyset \Big\}. \end{aligned}$$

Given $\mathcal{G}(N_j)$ the events $E(w)$, $w \in \mathcal{K}_{N_j}$ are independent and, by the lower bound in Lemma 4.2 for a suitable choice of integer n_j, they have probability exceeding

$$\exp\Big(-(N_j + m)\,(r + \tfrac{1}{j})\,(\mathfrak{a} - \theta - \tfrac{1}{N})\Big)$$

Hence the probability that among the vertices $w \in \mathcal{K}_{N_j}$ none satisfies $E(w)$ is less than

$$\Big(1 - \exp\Big(-(N_j + m)\,(r + \tfrac{1}{j})\,(\mathfrak{a} - \theta - \tfrac{1}{N})\Big)\Big)^{K_{N_j}} \;\overset{j\to\infty}{\longrightarrow}\; 0\,,$$

as $K_{N_j} \geq \exp\big(N_j\big(r(\mathfrak{a} - \theta) - \tfrac{1}{j}\big)\big)$. We infer that, almost surely on $\partial T(v)^*$, there exist infinitely many vertices $w \in T(v)^*$ such that $E(w)$ holds. Hence there exists a ray $\xi \in \partial T(v)^* \cap U(N)$ and we are done. ■

5 Other measures on the boundary of a Galton-Watson tree

The branching measure μ is not the only interesting measure on the boundary of a Galton-Watson tree. Liu and Rouault (1997) study the *harmonic measure* ν, which is also called the *visibility measure* or *equally–splitting measure.* It is obtained as the distribution of the random ray arising when a particle moves on the vertices of the tree starting at the root and choosing in each step independently and uniformly among the children of the current vertex.

They show that, almost surely, μ and ν are mutually singular. They also prove that, almost surely, for μ-almost every ray V,

$$\lim_{n\to\infty} \frac{\log \nu B(V, e^{-n})}{-n} = \frac{E[N \log N]}{EN},$$

whereas for ν-almost every ray V,

$$\lim_{n\to\infty} \frac{\log \nu B(V, e^{-n})}{-n} = E[\log N].$$

In particular the measure ν has carrying dimension $E[\log N]$, which is strictly smaller than the dimension of the branching measure, and ν is multifractal. Its multifractal spectrum seems to be unknown.

Lyons, Pemantle and Peres (1995,1996) study a whole family of *harmonic measures* on ∂T corresponding to random walks in which the particle is allowed to move backwards towards the root. They show that all these measures on ∂T are supported by a set of Hausdorff dimension strictly smaller than $\log \mathfrak{m}$. Again the multifractal structure of these measures is unknown, and represents a (probably rather hard) challenge for future research.

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